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Research Article

Interval Oscillation Theorems for the Weighted p-Sub-Laplacian Equation in the Heisenberg Group

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In this paper, we derive a Riccati-type inequality in the Heisenberg group H^n. Based on it, some oscillation criteria are established for the weighted p-sub-Laplacian equations in H^n. Our results generalize the oscillation theorems for p-sub-Laplacian equations in R^n to ones in H^n.

1. Introduction

In this paper, we consider the nonlinear degenerate elliptic equation in the Heisenberg group H^n:

$$\nabla_H \cdot \left(|\nabla_H u|^{p-2} A(z,t) \nabla_H u\right) + c(z,t)|u|^{p-2}u = 0, \quad (1)$$

where $p > 1$, $(z,t) \in \Omega$, Ω is an outer region in H^n, ∇_H denotes the Heisenberg gradient (see (19)), and $A(z,t)$ and $c(z,t)$ are to be specified later.

In the qualitative theory of nonlinear partial differential equations, one of the important problems is to determine whether or not solutions of the equations are oscillatory. For the second-order linear ordinary differential equation,

$$\left(r(t)x'(t)\right)' + q(t)x(t) = 0. \quad (2)$$

A classical result of the oscillation is the famous Fite-Wintner theorem which states that if $\lim_{t\longrightarrow\infty} q(t) = \infty$, then the solutions of (2) are oscillatory (see [1]). In [2], Kamenev studied the oscillatory behavior of the solutions of (2) under the assumption that $\lim_{t\longrightarrow\infty} q(t) < \infty$.

Soon after, Fite-Wintner's theorem and Kamenev's theorem were extended to various forms of second-order differential equations. In [3], by using the Riccati-type transformation, Noussair and Swanson extended Fite-Wintner's theorem to the equation:

$$\nabla\cdot(A(x)\nabla y) + q(x)f(y) = 0. \quad (3)$$

Usami [4] established Fite-Wintner-type theorem to the quasilinear elliptic equation in divergence form:

$$\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) + p(x)|u|^{p-2}u = 0. \quad (4)$$

Xu [5] and Zhuang and Wu [6] studied, respectively, the oscillation problem for the weighted elliptic equation:

$$\nabla\cdot\left(|\nabla u|^{p-2}A(x)\nabla u\right) + p(x)|u|^{p-2}u = 0. \quad (5)$$

For more results about differential equations, one can refer to [7–9] and references therein.

It knows that the p-Laplacian equations play a critical role in physical phenomena. We refer the readers to Díaz [10] for detailed references on physical background of the p-Laplacian equations. In this paper, we derive several oscillation criteria for the weighted p-sub-Laplacian equation in H^n. One of the difficulties is that there does not exist a good divergence formula in H^n as in R^n. In this paper, we overcome this difficulty.

Before stating our main results, we introduce some notations and notions. For positive constants a_1, a_2, we denote

$$\begin{aligned} G(a_1, a_2] &= \left\{(z, t) \in H^n : a_1 < |(z, t)|_H \le a_2\right\}, \\ G[a_1, a_2) &= \left\{(z, t) \in H^n : a_1 \le |(z, t)|_H < a_2\right\}, \\ G[a_1, a_2] &= \left\{(z, t) \in H^n : a_1 \le |(z, t)|_H \le a_2\right\}, \\ G[a_1, +\infty) &= \left\{(z, t) \in H^n : |(z, t)|_H \ge a_1\right\}, \end{aligned} \tag{6}$$

where $|(z, t)|_H$ denotes the norm in H^n (see (22)). A domain Ω is called the outer region in H^n if there exists a positive constant a_0 such that $G[a_0, +\infty) \subset \Omega$. Let us restrict our attention to the nontrivial solution $u(z, t)$ of (1), that is, to the solution $u(z, t)$ satisfying

$$\sup \left\{|u(z, t)| : (z, t) \in \Omega\right\} > 0. \tag{7}$$

A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

Now, we give a definition.

Definition 1. Let

$$D_0 = \left\{(r, s) \in R^2 : r > s \ge a_0\right\}, D = \left\{(r, s) \in R^2 : r \ge s \ge a_0\right\}. \tag{8}$$

We call that F belongs to the function class Ψ, if $F \in C(D, R)$ and there exist $f_1, f_2 \in C(D, R)$ and $\phi \in C^1([a_0, +\infty), R^+)$ such that

(F_1) $F(r, r) = 0$ for $r \ge a_0$ and $F(r, s) > 0$ for all $(r, s) \in D_0$

(F_2) $(\partial/\partial r)(F(r, s)) + (\phi'(r)/\phi(r))F(r, s) = f_1(r, s)(F(r, s))^{(p-1)/p}$

(F_3) $(\partial/\partial s)(F(r, s)) + (\phi'(s)/\phi(s))F(r, s) = f_2(r, s)(F(r, s))^{(p-1)/p}$

In this paper, we always assume that the following conditions are satisfied.

(C_1) The coefficient matrix $A(z, t) = (a_{ij}(z, t))_{2n\times 2n}$ is a real symmetric positive definite matrix function (i.e., it is the ellipticity condition in R^{2n}) with $a_{ij} \in C^2_{\text{loc}}(\Omega, R)$, the smallest (necessarily positive) eigenvalue of $A(z, t)$ is denoted by $\lambda_{\min}(z, t)$, and there exists a function $\rho(r) \in C(R^+, R^+)$ such that

$$\rho(r) \le \min_{|(z,t)|_H = r} \frac{\lambda_{\min}(z, t)}{|A(z, t)|^{p/(p-1)}}, \quad r > a_0, \tag{9}$$

where $|A(z, t)| = \left(\sum_{i,j=1}^{2n} a_{ij}^2(z, t)\right)^{1/2}$.

(C_2) $c(z, t) \in C^2_{\text{loc}}(\Omega, R)$.

For convenience, denote

$$\theta(r) = \phi(r) \int_{S_H(o,r)} c(z, t)\, dH, \tag{10}$$

where $S_H(o, r)$ denotes the sphere in H^n with the center $o = (0, 0)$ and the radius r and dH denotes the $2n$-dimensional Hausdorff measure in R^{2n+1} (see [11]):

$$g(r) = \frac{1}{\rho(r)} \left(\wp\alpha_\wp r^{2n+1} \phi(r)\right)^{1/(p-1)}, \tag{11}$$

where $\wp = 2n + 2$ is the homogeneous dimension of H^n and $\wp\alpha_\wp$ ($\alpha_\wp$ is a constant) denotes the area of unit sphere $S_H(o, 1)$ in H^n.

One of the main results is the following.

Theorem 2. *Assume that for any $T \ge a_0$, there exist $T \le a < c < b$ and $F \in \Psi$ such that*

$$\begin{aligned} &\frac{1}{F(c, a)} \int_a^c F(s, a)\theta(s)\, ds + \frac{1}{F(b, c)} \int_c^b F(b, s)\theta(s)\, ds \\ &\quad > \frac{1}{p^p} \frac{1}{F(c, a)} \int_a^c \left(4s^2 + 1\right)^{p/2} g^{p-1}(s) |f_1(s, a)|^p\, ds \\ &\quad + \frac{1}{p^p} \frac{1}{F(b, c)} \int_c^b \left(4s^2 + 1\right)^{p/2} g^{p-1}(s) |f_2(b, s)|^p\, ds. \end{aligned} \tag{12}$$

Then, (1) is oscillatory.

Denote

$$R(r) = \int_{a_0}^r \frac{1}{(4s^2 + 1)^{p/2(p-1)} g(s)}\, ds, \text{ for } r \ge a_0, \tag{13}$$

and take

$$\begin{aligned} F(r, s) &= (R(r) - R(s))^\lambda, \text{ for } r \ge s \ge a_0, \\ \phi(r) &= 1, \end{aligned} \tag{14}$$

in (10), where $\lambda > \max(1, p - 1)$ is a constant. The following is a Kamenev-type oscillation criterion.

Theorem 3. *Assume $R(r) \longrightarrow \infty (r \longrightarrow \infty)$. If for each $l \ge a_0$, there exist $\lambda > \max(1, p - 1)$ such that*

$$\limsup_{r\longrightarrow\infty} \frac{1}{R^{\lambda-p+1}(r)} \int_l^r (R(s) - R(l))^\lambda \theta(s)\, ds > \frac{\lambda^p}{p^p(\lambda - p + 1)}, \tag{15}$$

$$\limsup_{r\longrightarrow\infty} \frac{1}{R^{\lambda-p+1}(r)} \int_l^r (R(r) - R(s))^\lambda \theta(s)\, ds > \frac{\lambda^p}{p^p(\lambda - p + 1)}. \tag{16}$$

Then, (1) is oscillatory.

The paper is organized as follows. In Section 2, we collect some well-known results for the Heisenberg group and introduce two lemmas. Section 3 is devoted to the proofs of the Riccati-type inequality. The proofs of Theorems 2 and 3 are given in Section 4.

2. Preliminaries

The Heisenberg group H^n is R^{2n+1} (or $C^n \times R$) endowed with the group law $\circ$ defined by

$$\bar{\xi} \circ \xi = \left(x + \bar{x}, y + \bar{y}, t + \bar{t} + 2 \sum_{i=1}^{n} (x_i \bar{y}_i - y_i \bar{x}_i) \right), \qquad (17)$$

where $\xi = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, t) \coloneqq (x, y, t) \coloneqq (z, t) \in R^{2n} \times R, \bar{\xi} = (\bar{z}, \bar{t})$. The group H^n plays the important roles as R^n in conformal geometry, geometry of several complexes, and harmonic analysis (e.g., see Folland and Stein in [12]).

The left invariant vector fields on H^n are of the form

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, j = 1, 2, \cdots, n, T = \frac{\partial}{\partial t}. \qquad (18)$$

The family $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ satisfies Hörmander's rank condition (see [13]). The Heisenberg gradient of a smooth function u is defined by

$$\nabla_H u = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u). \qquad (19)$$

The divergence of a smooth vector value function $F = (F_1, \cdots, F_{2n})$ on H^n is defined by

$$\nabla_H \cdot F = X_1 F_1 + \cdots + X_n F_n + Y_1 F_{n+1} + \cdots + Y_n F_{2n}. \qquad (20)$$

For $F = (F_1, \cdots, F_{2n+1})$, the usual divergence div F on R^{2n+1} is

$$\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n} + \frac{\partial F_{n+1}}{\partial y_1} + \cdots + \frac{\partial F_{2n}}{\partial y_n} + \frac{\partial F_{2n+1}}{\partial t}. \qquad (21)$$

The norm $|\xi|_H$ for $\xi \in H^n$ is

$$|\xi|_H = \left(\left(x^2 + y^2 \right)^2 + t^2 \right)^{1/4}. \qquad (22)$$

With the norm, the distance between two points ξ and η in H^n is defined by

$$d_H(\xi, \eta) = \left| \eta^{-1} \circ \xi \right|_H, \qquad (23)$$

where η^{-1} denotes the inverse of η with respect to $\circ$, that is, $\eta^{-1} = -\eta$.

The sphere of radius $r > 0$ centered at the origin $o = (0, 0)$ of H^n is the set:

$$S_H(o, r) = \{ \xi \in H^n : d_H(\xi, o) = r \}, \qquad (24)$$

and the open ball of radius $r > 0$ centered at o is the set:

$$B_H(o, r) = \{ \xi \in H^n : d_H(\xi, o) < r \}. \qquad (25)$$

From [11], we know that the area of $S_H(o, r)$ is

$$|S_H(o, r)| = \wp \alpha_\wp r^{2n+1}, \qquad (26)$$

where $\alpha_\wp$ is the volume of $B_H(o, 1)$ [14]. For simplicity, we will denote $B_H(o, r)$ and $S_H(o, r)$ by B_r and S_r, respectively.

Now, we first introduce two well-known lemmas.

Lemma 4 (refer to [15]). *If α and β are nonnegative constants and $q > 1$, then*

$$\alpha^q - q \alpha \beta^{q-1} + (q-1) \beta^q \geq 0. \qquad (27)$$

Lemma 5 (the divergence formula in H^n [12]). *Let Ω_0 be a bounded domain in H^n with C^1 boundary $\partial \Omega_0$ and ν denote the unit outward normal to $\partial \Omega_0$. For any $C^1(\Omega_0)$ vector field $V = (V_1, \cdots, V_{2n})$, we have*

$$\int_{\Omega_0} \nabla_H \cdot V dz dt = \int_{\partial \Omega_0} M V \cdot \nu dH, \qquad (28)$$

where

$$M = \begin{pmatrix} I_n & 0 \\ 0 & I_n \\ 2y & -2x \end{pmatrix}_{(2n+1) \times 2n}. \qquad (29)$$

I_n is the identity matrix of R^n.

The following lemma plays a critical role in proving the Riccati-type inequality. The proof is similar to Theorem 2.3 in [14], and we omit it.

Lemma 6. *For any $C^1(B_r)$ vector field $V = (V_1, \cdots, V_{2n})$ and $p > 1$, we have*

$$\| M V \cdot \nu \|^2_{L^1(S_r)} \leq \left(\wp \alpha_\wp r^{2n+1} \right)^{2/p} \left(4r^2 + 1 \right) \| V \|^2_{L^{p/(p-1)}(S_r)}. \qquad (30)$$

3. A Riccati-Type Inequality

In this section, we establish a Riccati-type inequality and then prove two lemmas.

Lemma 7 (type inequality). *Suppose that $u(z, t)$ is the nontrivial solution of (1) with $u(z, t) > 0$ for $(z, t) \in G[c, b)$. Let*

$$v = v(z, t) = \log |u(z, t)|, \qquad (31)$$

$$W(z, t) = |\nabla_H v|^{p-2} A(z, t) \nabla_H v, (z, t) \in G[c, b), \qquad (32)$$

$$Z(r) = \phi(r) \int_{S_H(o,r)} M W(z, t) \cdot \nu dH, (z, t) \in G[c, b). \qquad (33)$$

Then,

$$Z'(r) \le \frac{\phi'(r)}{\phi(r)} Z(r) - \theta(r) - \frac{p-1}{g(r)} |Z(r)|^{p/(p-1)} \left(4r^2+1\right)^{p/2(1-p)}. \tag{34}$$

Proof. It easily knows that

$$X_i v = \frac{\partial v}{\partial x_i} + 2y_i \frac{\partial v}{\partial t} = \frac{1}{u}\frac{\partial u}{\partial x_i} + 2y_i \frac{1}{u}\frac{\partial u}{\partial t} = \frac{1}{u} X_i u, \tag{35}$$

and similarly,

$$Y_i v = \frac{1}{u} Y_i u. \tag{36}$$

Hence,

$$\nabla_H v = \frac{1}{u} \nabla_H u. \tag{37}$$

Combining (1) and (32), we have

$$\begin{aligned}
\nabla_H \cdot W(z,t) &= \nabla_H \cdot \left(|\nabla_H v|^{p-2} A(z,t) \nabla_H v\right) \\
&= \nabla_H \cdot \left(\frac{1}{u^{p-1}} |\nabla_H u|^{p-2} A(z,t) \nabla_H u\right) \\
&= \frac{1}{u^{p-1}} \nabla_H \cdot \left(|\nabla_H u|^{p-2} A(z,t) \nabla_H u\right) + |\nabla_H u|^{p-2} A(z,t) \nabla_H u \nabla_H \left(\frac{1}{u^{p-1}}\right) \\
&= \frac{1}{u^{p-1}} \left(-c(z,t)|u|^{p-2}u\right) - (p-1)\frac{1}{u^p} (\nabla_H u)^T |\nabla_H u|^{p-2} A(z,t) \nabla_H u \\
&= -c(z,t) - (p-1)|\nabla_H v|^{p-2} (\nabla_H v)^T A(z,t) \nabla_H v,
\end{aligned} \tag{38}$$

that is,

$$\nabla_H \cdot W(z,t) + c(z,t) + (p-1)|\nabla_H v|^{p-2} (\nabla_H v)^T A(z,t) \nabla_H v = 0. \tag{39}$$

By integrating (39) over $S_H(o,r)(r \ge c)$, it follows

$$\begin{aligned}
&\int_{S_H(o,r)} \nabla_H \cdot W(z,t) dH + \int_{S_H(o,r)} c(z,t) dH \\
&\quad + (p-1) \int_{S_H(o,r)} |\nabla_H v|^{p-2} (\nabla_H v)^T A(z,t) \nabla_H v dH = 0.
\end{aligned} \tag{40}$$

Using (28) and (10), it implies

$$\begin{aligned}
Z'(r) &= \phi'(r) \int_{S_H(o,r)} MW(z,t) \cdot \nu dH \\
&\quad + \phi(r) \int_{\partial S_H(o,r)} MW(z,t) \cdot \nu d\sigma \\
&= \frac{\phi'(r)}{\phi(r)} Z(r) + \phi(r) \int_{S_H(o,r)} \nabla_H \cdot W(z,t) dH \\
&= \frac{\phi'(r)}{\phi(r)} Z(r) - \theta(r) - (p-1)\phi(r) \\
&\quad \cdot \int_{S_H(o,r)} |\nabla_H v|^{p-2} (\nabla_H v)^T A(z,t) \nabla_H v dH.
\end{aligned} \tag{41}$$

For $W(z,t)$ in (32), we have

$$|W(z,t)| = \left||\nabla_H v|^{p-2} A(z,t) \nabla_H v\right| \le |\nabla_H v|^{p-1} |A(z,t)|. \tag{42}$$

It yields

$$|\nabla_H v| \ge \left(\frac{|W(z,t)|}{|A(z,t)|}\right)^{1/(p-1)}. \tag{43}$$

In view of (C_1), we have

$$\begin{aligned}
|\nabla_H v|^{p-2} (\nabla_H v)^T A(z,t) \nabla_H v &\ge |\nabla_H v|^{p-2} \lambda_{\min}(z,t) |\nabla_H v|^2 \\
&= \lambda_{\min}(z,t) |\nabla_H v|^p \\
&\ge \lambda_{\min}(z,t) \left(\frac{|W(z,t)|}{|A(z,t)|}\right)^{p/(p-1)} \\
&\ge \rho\left(|(z,t)|_H\right) |W(z,t)|^{p/(p-1)}.
\end{aligned} \tag{44}$$

By (30) and (33), we have

$$\begin{aligned}
\left|\frac{Z(r)}{\phi(r)}\right| &= \left|\int_{S_H(o,r)} MW(z,t) \cdot \nu dH\right| \\
&\le \left(\wp \alpha_\wp r^{2n+1}\right)^{1/p} \left(4r^2+1\right)^{1/2} \left(\int_{S_H(o,r)} |W|^{p/(p-1)} dH\right)^{(p-1)/p}.
\end{aligned} \tag{45}$$

By combining (41), (44), (45), and (11), it yields

$$\begin{aligned}
Z'(r) &\le \frac{\phi'(r)}{\phi(r)} Z(r) - \theta(r) - (p-1)\phi(r)\rho(r) \\
&\quad \cdot \left|\frac{Z(r)}{\phi(r)}\right|^{p/(p-1)} \left(\wp \alpha_\wp r^{2n+1}\right)^{1/(1-p)} \left(4r^2+1\right)^{p/2(1-p)} \\
&= \frac{\phi'(r)}{\phi(r)} Z(r) - \theta(r) - \frac{p-1}{g(r)} |Z(r)|^{p/(p-1)} \left(4r^2+1\right)^{p/2(1-p)}.
\end{aligned} \tag{46}$$

The proof is complete. □

Using Lemma 7, we have the following.

Lemma 8. *Suppose that $u(z,t)$ is the nontrivial solution of (1) with $u(z,t)>0$ for $(z,t)\in G[c,b)$ and $F\in\Psi$. Let $W(z,t)$ and $Z(r)$ be the same as Lemma 7; then,*

$$\frac{1}{F(b,c)}\int_c^b F(b,s)\theta(s)ds \le Z(c)+\frac{1}{p^p F(b,c)} \cdot \int_c^b \left(4s^2+1\right)^{p/2} g^{p-1}(s)\left|f_2(b,s)\right|^p ds. \tag{47}$$

Proof. Changing r to s in (34), multiplying (34) by $F(r,s)$, and integrating from c to r, we have in view of (F_1) and (F_3) that

$$\begin{aligned}\int_c^r F(r,s)\theta(s)ds &\le -\int_c^r Z'(s)F(r,s)ds+\int_c^r \frac{\phi'(s)}{\phi(s)}F(r,s)Z(s)ds\\ &\quad-(p-1)\int_c^r \frac{F(r,s)}{g(s)}|Z(s)|^{p/(p-1)}\left(4s^2+1\right)^{p/2(1-p)}ds\\ &\le F(r,c)Z(c)+\int_c^r \left|f_2(r,s)\right|\left|F(r,s)\right|^{(p-1)/p}|Z(s)|ds\\ &\quad-(p-1)\int_c^r \frac{F(r,s)}{g(s)}|Z(s)|^{p/(p-1)}\left(4s^2+1\right)^{p/2(1-p)}ds.\end{aligned} \tag{48}$$

Take

$$\begin{aligned}\alpha &= \left|(p-1)F(r,s)\right|^{(p-1)/p}\frac{|Z(s)|}{\left(4r^2+1\right)^{1/2}g^{(p-1)/p}(s)},\\ \beta &= \frac{(p-1)^{(p-1)/p}\left(4r^2+1\right)^{(p-1)/2}g^{(p-1)^2/p}(s)\left|f_2(r,s)\right|^{p-1}}{p^{p-1}},\\ q &= \frac{p}{p-1}.\end{aligned} \tag{49}$$

Then, by (27), we get

$$\begin{aligned}&\left|f_2(r,s)\right|\left|F(r,s)\right|^{(p-1)/p}|Z(s)|-(p-1)\frac{F(r,s)}{\left(4s^2+1\right)^{p/2(p-1)}g(s)}|Z(s)|^{p/(p-1)}\\ &\quad\le \frac{\left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_2(r,s)\right|^p}{p^p}.\end{aligned} \tag{50}$$

By combining (48), it shows

$$\int_c^r F(r,s)\theta(s)ds \le F(r,c)Z(c) + \frac{1}{p^p}\int_c^r \left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_2(r,s)\right|^p ds. \tag{51}$$

By letting $r\longrightarrow b^-$ and dividing both sides by $F(b,c)$, it follows (47). □

Lemma 9. *Suppose that $u(z,t)$ is the nontrivial solution of (1) with $u(z,t)>0$ for $(z,t)\in G[a,c)$ and $F\in\Psi$. Let $W(z,t)$ and $Z(r)$ be similarly with Lemma 7 for $(z,t)\in G[a,c)$; then,*

$$\frac{1}{F(c,a)}\int_a^c F(s,a)\theta(s)ds \le -Z(c)+\frac{1}{p^p F(c,a)} \cdot \int_a^c \left(4s^2+1\right)^{p/2} g^{p-1}(s)\left|f_1(s,a)\right|^p ds. \tag{52}$$

Its proof is similar to that of Lemma 8, so we omit it here.

4. Proofs of the Main Results

The following lemma is useful for proving Theorem 2.

Lemma 10. *If there exist $c\in(a,b)$ and $F\in\Psi$ such that (12) holds, then every nontrivial solution $u(z,t)$ of (1) has at least one zero in $G[a,b]$.*

Proof. Suppose that the statement is incorrect; without loss generality, we may assume that there exists a solution $u(z,t)$ of (1) such that

$$u(z,t)>0 \text{ for } (z,t)\in G[a,b]. \tag{53}$$

From Lemmas 8 and 9, it implies that (47) and (52) hold. Using them, we have

$$\begin{aligned}&\frac{1}{F(c,a)}\int_a^c F(s,a)\theta(s)ds+\frac{1}{F(b,c)}\int_c^b F(b,s)\theta(s)ds\\ &\le \frac{1}{p^p}\frac{1}{F(c,a)}\int_a^c \left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_1(s,a)\right|^p ds\\ &\quad+\frac{1}{p^p}\frac{1}{F(b,c)}\int_c^b \left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_2(b,s)\right|^p ds,\end{aligned} \tag{54}$$

which contradicts to (12). Thus, the claim is true. □

Now, we give the following.

Proof of Theorem 1. Take the sequence

$$\{T_i\}\subset[a_0,+\infty) \text{ with } \lim_{i\longrightarrow\infty} T_i=\infty. \tag{55}$$

By assumptions, we see that for each $i\in N$, there exist $a_i, b_i, c_i \in R$ such that $T_i\le a_i<c_i<b_i$ and (12) holds. In view to Lemma 10, we conclude that every nontrivial solution $u(z,t)$ of (1) has at least one zero (z,t) in $G[a_i,b_i]$. By noting $|(z,t)|_H\ge a_i\ge T_i$, $i\in N$, it follows that every solution has arbitrarily large zeros. Hence, (1) is oscillatory.

As an immediate consequence of Theorem 2, the following result is true. □

Corollary 11. *If (12) in Theorem 2 is replaced by*

$$\limsup_{r\longrightarrow\infty}\int_l^r \left(F(s,l)\theta(s)-\frac{\left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_1(s,l)\right|^p}{p^p}\right)ds>0, \tag{56}$$

$$\limsup_{r\longrightarrow\infty}\int_l^r \left(F(r,s)\theta(s)-\frac{\left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_2(r,s)\right|^p}{p^p}\right)ds>0, \tag{57}$$

for each sufficient large $l\geq a_0$; then, (1) is oscillatory.

Proof. For any $T\geq a_0$, let $l=T$. We choose $a=l$ in (56); then, there exists $c>a$ such that

$$\int_a^c \left(F(s,a)\theta(s)-\frac{\left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_1(s,a)\right|^p}{p^p}\right)ds>0. \tag{58}$$

By letting $c=l$ in (57), there exists $b>c$ such that

$$\int_c^b \left(F(b,s)\theta(s)-\frac{\left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_2(b,s)\right|^p}{p^p}\right)ds>0. \tag{59}$$

By combining (58) and (59), it yields (12). The conclusion is proven from Theorem 2. □

Proof of Theorem 2. From Definition 1 and the definitions of $F(r,s)$ and $R(r)$, we get

$$\begin{aligned}
f_1(r,s)&=\lambda(R(r)-R(s))^{(\lambda-p)/p}\frac{1}{\left(4r^2+1\right)^{p/2(p-1)}g(r)},\\
f_2(r,s)&=-\lambda(R(r)-R(s))^{(\lambda-p)/p}\frac{1}{\left(4s^2+1\right)^{p/2(p-1)}g(s)}.
\end{aligned} \tag{60}$$

Thus, it follows

$$\begin{aligned}
&\int_l^r \left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_1(s,l)\right|^p ds\\
&\quad=\int_l^r \lambda^p(R(s)-R(l))^{\lambda-p}\frac{1}{\left(4s^2+1\right)^{p/2(p-1)}g(s)}ds\\
&\quad=\frac{\lambda^p}{\lambda-p+1}(R(r)-R(l))^{\lambda-p+1},
\end{aligned}$$

$$\begin{aligned}
&\int_l^r \left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_2(r,s)\right|^p ds\\
&\quad=\int_l^r \lambda^p(R(r)-R(s))^{\lambda-p}\frac{1}{\left(4s^2+1\right)^{p/2(p-1)}g(s)}ds\\
&\quad=\frac{\lambda^p}{\lambda-p+1}(R(r)-R(l))^{\lambda-p+1}.
\end{aligned} \tag{61}$$

Noting $\lim_{r\longrightarrow\infty}R(r)=\infty$, we see

$$\lim_{r\longrightarrow\infty}\frac{1}{p^pR^{\lambda-p+1}(r)}\int_l^r \left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_1(s,l)\right|^p ds=\frac{\lambda^p}{p^p(\lambda-p+1)}, \tag{62}$$

$$\lim_{r\longrightarrow\infty}\frac{1}{p^pR^{\lambda-p+1}(r)}\int_l^r \left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_2(r,s)\right|^p ds=\frac{\lambda^p}{p^p(\lambda-p+1)}. \tag{63}$$

By combining (63) and (15), it yields

$$\begin{aligned}
&\limsup_{r\longrightarrow\infty}\frac{1}{R^{\lambda-p+1}(r)}\int_l^r \left[(R(s)-R(l))^\lambda\theta(s)-\frac{1}{p^p}\left(4s^2+1\right)^{p/2}g^{p-1}(s)\left|f_1(s,l)\right|^p\right]ds\\
&\quad=\limsup_{r\longrightarrow\infty}\frac{1}{R^{\lambda-p+1}(r)}\int_l^r (R(s)-R(l))^\lambda\theta(s)ds-\frac{\lambda^p}{p^p(\lambda-p+1)}>0,
\end{aligned} \tag{64}$$

which implies (56). Similarly, (57) holds by combining (63) and (16). From Corollary 11, (1) is oscillatory. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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