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Research Article

Three Types of Distributional Chaos for a Sequence of Uniformly Convergent Continuous Maps

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Let $(h_s)_{s=1}^{\infty}$ be a sequence of continuous maps on a compact metric space W which converges uniformly to a continuous map h on W. In this paper, some equivalence conditions or necessary conditions for the limit map h to be distributional chaotic are obtained (where distributional chaoticity includes distributional chaotic in a sequence, distributional chaos of type 1 (DC1), distributional chaos of type 2 (DC2), and distributional chaos of type 3 (DC3)).

1. Introduction

In this paper, a topological dynamical system (shortly, TDS) is a pair (W, h), where $h : W \longrightarrow W$ is a continuous surjective map that acts on a compact metric space W with a metric e. And let $\mathbb{Z}^+$ be the set of nonnegative integers.

To find conditions that assure the preservation of any chaotic property under limit operations is an interesting problem (see [1–11]). In [8], the author proved that if $h_s(s \in \{1, 2, \cdots\})$ are continuous transitive functions on a metric space (W, e), h_s converge uniformly to a function h, and then, a few sufficient conditions for the limit function h to be topologically transitive were presented. In [5], the authors discussed the dynamical behaviour of the uniform limit of a sequence of continuous maps on a compact metric space which satisfy (topological) transitivity or other related properties and presented some conditions different from [8] for the transitivity of a limit map. In [12], the limit behaviour of sequences with the form $h_s \circ \cdots \circ h_1(a)$, $a \in [0, 1]$ was studied; the author considered whether the simplicity (respectively, chaoticity) of h implies the simplicity (respectively, chaoticity) of $h_{1,\infty}$ (where $h_{1,\infty} = (h_s)_{s=1}^{\infty}$ is a sequence of continuous interval maps converging uniformly to a continuous map h). More recently, [4] considered nonautonomous discrete dynamical systems $h_{1,\infty}$, which were given by surjective continuous map sequence $h_s : I \longrightarrow I$ converged uniformly to a map $h : [0, 1] \longrightarrow [0, 1]$. It is obtained that the full Lebesgue measure of a distributional scrambled set of the nonautonomous system cannot guarantee the existence of distributional chaos of the limit map. There exists a nonautonomous system with an arbitrarily small distributional scrambled set which converges to a map that is distributional chaotic almost everywhere. As one knows, sensitivity property characterizes the unpredictability of chaotic phenomenon in a system and it is one of the essential conditions of various definitions of a chaotic system. So, when is a system sensitive? This question has gained some attention in more recent papers (see [10, 13–16]). A TDS (W, h) is sensitive if for any region E of the phase space W there are two points in E and some $s \in \{0, 1, \cdots\}$ satisfying that the sth iterate of the two points under the map h is significantly separated. The size of the set of all $s \in \mathbb{Z}^+$ satisfying that this significant separation or sensitivity happens can be considered a measure of how sensitive the dynamical system is.

In particular, if this set is relatively thin with arbitrarily large gaps between consecutive entries, then we can consider the dynamical system as practically nonsensitive!

In [10], the authors obtained an equivalence condition under which the uniform limit h is sensitive. In [7], we obtain an equivalence condition for that the uniform map h is topologically transitive (resp., syndetically transitive, topologically weak mixing, and topological mixing). Moreover, necessary conditions for the uniform map h to be sensitive or cofinitely sensitive or multisensitive were given. In [17], we present the correct proofs of Theorems 4–8 in [7]. Moreover, for a continuous map sequence $(h_s)_{s=1}^{\infty}$ on a compact metric space W converges uniformly to a continuous map $h : W \longrightarrow W$, we present an equivalence condition under which the uniform map h is syndetically sensitive or cofinitely sensitive or multisensitive or ergodically sensitive, and a sufficient condition under which the uniform map h is totally transitive or topologically weak mixing. In [18], we gave an equivalence condition under which the uniform limit map is $\mathscr{F}$-transitive or weakly $\mathscr{F}$-sensitive or $\mathscr{F}$-sensitive or $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive and a necessary condition for the uniform limit map to be weakly $\mathscr{F}$-sensitive or $\mathscr{F}$-sensitive or $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive.

In this paper, on a compact metric space W, for a continuous map sequence $(h_n)_{n=1}^{\infty}$ which converges uniformly to a continuous map h, we obtain equivalence conditions under which the uniform map h is distributional chaotic or distributional chaotic in a sequence or DC1 or DC2 or DC3 and necessary conditions for the uniform map h to be distributional chaotic or distributional chaotic in a sequence or DC1 or DC2 or DC3.

In Section 2, some concepts are recalled. Distributional chaotic properties are obtained and proved in Section 3.

2. Preliminaries

For any dynamical systems (W_1, e_1) and (W_2, e_2), define $e_{\infty}(e_1, e_2) = \sup_{a \in W} e(e_1(a), e_2(a))$, where W is a perfect metric space (see [8]).

Let $\{q_j\}_{j=1}^{\infty}$ be a sequence of integers such that $q_j > 0$ and $q_{j+1} > q_j$ for any $j \in \{1, 2, \cdots\}$. For a continuous map $h : W \longrightarrow W$ of a metric space W with metric e, any $a, b \in W$ and any $t > 0$, let

$$\phi_{ab}\left(t, h, \left\{q_j\right\}\right) = \liminf_{m \longrightarrow \infty} \frac{1}{m} \sum_{j=1}^{m} \chi_{[0,t)}\left(e(h^{q_j}(a), e(h^{q_j}(b))\right),$$
$$\phi^*_{ab}\left(t, h, \left\{q_j\right\}\right) = \limsup_{m \longrightarrow \infty} \frac{1}{m} \sum_{j=1}^{m} \chi_{[0,t)}\left(e(h^{q_j}(a), e(h^{q_j}(b))\right), \tag{1}$$

where $\chi_A(y)$ is 1 if $y \in A$ and 0 otherwise. Let $E \subset W$ and $a, b \in E$ with $a \neq b$. The set E is said to be a distributional chaotic set in a sequence, and a, b are said to be a pair of points displaying distributional chaos in a sequence if

(1) $\phi_{ab}(p, h, \{q_j\}_{j=1}^{\infty}) = 0$, for some $p > 0$

(2) $\phi^*_{ab}(t, h, \{q_j\}_{j=1}^{\infty}) = 1$, for any $t > 0$

A continuous map $h : W \longrightarrow W$ is said to be distributional chaotic in a sequence if it has a distributional chaotic set in a sequence which is uncountable. In particular, a continuous map $h : W \longrightarrow W$ is said to be distributional chaotic if it is distributional chaotic in the sequence of positive integers.

Let $\phi_{ab}(t, h) = \phi_{ab}(t, h, \{1, 2, \cdots\})$ and $\phi^*_{ab}(t, h) = \phi^*_{xy}(t, h, \{1, 2, \cdots\})$ for any $t \geq 0$. For a continuous map h on a metric space W, if there is an uncountable set $E \subset W$, for any $a, b \in E$ with $a \neq b$, $\phi^*_{ab}(t, h) \equiv 1$ for every $t > 0$ and $\phi_{ab}(t_0, h) = 0$ for some $t_0 > 0$, or $\phi^*_{ab}(t, h) \equiv 1$ for every $t > 0$ and $\phi_{xy}(t_0, h) < \phi^*_{ab}(t_0, h)$ for some $t_0 > 0$, or $\phi_{ab}(t, h) < \phi^*_{ab}(t, h)$ for every $t > 0$, then we say that h is DC1, DC2, or DC3, respectively (see [19]).

3. Main Results

In all theorems in this section, (W, e) is a metric space. (h_n) be a sequence of continuous maps which converges uniformly to h on W. And it is always assumed that $\lim_{n \longrightarrow \infty} e_{\infty}(h_n^n, h^n) = 0$.

Theorem 1. *Let $\{q_j\}_{j=1}^{\infty}$ be a strictly infinitely increasing sequence of positive integers. Then, E is a distributional chaotic set in the sequence $\{q_j\}_{j=1}^{\infty}$ for h if and only if the following conditions are satisfied:*

(1) For any given $a, b \in E$ with $a \neq b$, $\phi_{ab}(p, (h_n^n), \{q_j\}_{j=1}^{\infty}) = 0$ for some $p > 0$, where

$$\phi_{ab}\left(t, (h_n^n), \left\{q_j\right\}_{j=1}^{\infty}\right) = \liminf_{n \longrightarrow \infty} \frac{1}{n} \sharp\left(\{1, 2, \cdots, n\} \cap \left\{m \in \mathbb{N} : e\left(h_{q_m}^{q_m}(a), h_{q_m}^{q_m}(b)\right) < t\right\}\right), \tag{2}$$

for any $t > 0$.

(2) For any given $a, b \in E$ with $a \neq b$, $\phi^*_{ab}(t, (h_n^n), \{q_j\}_{j=1}^{\infty}) = 1$ for any $t > 0$, where

$$\phi^*_{ab}\left(t, (f_n^n), \left\{q_j\right\}_{j=1}^{\infty}\right) = \limsup_{n \longrightarrow \infty} \frac{1}{n} \sharp\left(\{1, 2, \cdots, n\} \cap \left\{m \in \mathbb{N} : e\left(h_{q_m}^{q_m}(a), h_{q_m}^{q_m}(b)\right) < t\right\}\right), \tag{3}$$

for any $t > 0$, where $\sharp(\cdot)$ is the cardinal number of a set.

Proof (necessity). Assume that D is a distributional chaotic set in the sequence $\{q_j\}_{j=1}^{\infty}$ for h.

On the one hand, for any given $a, b \in D : a \neq b$, $\phi^{*}_{ab}((1/3)t, h, \{q_j\}_{j=1}^{\infty}) = 1$ for any $t > 0$. By the assumption that

$$\lim_{n\longrightarrow\infty} e_{\infty}(h_n^n, h^n) = 0, \tag{4}$$

for any given $t > 0$, there is a positive integer n_t such that

$$e_{\infty}(h_m^m, h^m) < \frac{1}{3}t, \tag{5}$$

for any $m \geq n_t$. Then, we have

$$\begin{aligned} e(h_m^m(a), h_m^m(b)) &\leq e(h_m^m(a), h^m(a)) + e(h^m(a), h^m(b)) \\ &\quad + e(h_m^m(b), h^m(b)) < \frac{1}{3}t + \frac{1}{3}t + \frac{1}{3}t < t, \end{aligned} \tag{6}$$

for any $m \geq n_t$. This shows that $\phi^{*}_{ab}(t, (h_n^n), \{q_j\}_{j=1}^{\infty}) = 1$ for any $t > 0$. Hence, condition (1) holds.

On the other hand, for any given $a, b \in D : a \neq b$, $\phi_{ab}(3p, h, \{q_j\}_{j=1}^{\infty}) = 0$ for some $p > 0$. This implies that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp(\{j : e(h^{q_j}(a), h^{q_j}(b)) \geq 3p, 1 \leq j \leq n\}) = 1. \tag{7}$$

By (4), for the above $p > 0$, there is an integer $n_p > 0$ such that

$$\begin{aligned} |e(h_n^n(a), h_n^n(b)) - e(h^n(a), h^n(b))| &\leq e(h_n^n(a), h^n(a)) \\ &\quad + e(h_n^n(b), h^n(b)) < p + p = 2p, \end{aligned} \tag{8}$$

for any $a, b \in W$ and any integer $n \geq n_p$. Therefore,

$$e(h^n(a), h^n(b)) < e(h_n^n(a), h_n^n(b)) + 2p, \tag{9}$$

for any integer $n \geq n_p$. That is,

$$e(h_n^n(a), h_n^n(b)) > e(h^n(a), h^n(b)) - 2p > p, \tag{10}$$

for any integer $n \geq n_p$. This implies that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp\left(\left\{j : e\left(h_{q_j}^{q_j}(a), h_{q_j}^{q_j}(b)\right) \geq 3p, 1 \leq j \leq n\right\}\right) = 1. \tag{11}$$

That is, $\phi_{ab}(p, (h_n^n), \{q_j\}_{j=1}^{\infty}) = 0$ for the above $p > 0$. So, condition (2) is true.

(Sufficiency) suppose that conditions (1) and (2) are true. From the above argument, we know that

$$\begin{aligned} e(h^m(a), h^m(b)) &\leq e(h_m^m(a), h^m(a)) + e(h_m^m(a), h_m^m(b)) \\ &\quad + e(h_m^m(b), h^m(b)) < \frac{1}{3}t + \frac{1}{3}t + \frac{1}{3}t < t, \end{aligned} \tag{12}$$

for any $m \geq n_t$, any $a, b \in D : a \neq b$, and any $t > 0$.

Since $\phi^{*}_{ab}((1/3)t, (h_n^n), \{q_j\}_{j=1}^{\infty}) = 1$ for any $t > 0$ and for any $a, b \in D : a \neq b$, $\phi^{*}_{ab}(t, h, \{q_j\}_{j=1}^{\infty}) = 1$ for any $t > 0$ and any $a, b \in D : a \neq b$. By hypothesis, for any given $a, b \in D : a \neq b$, $\phi_{ab}(3p, (h_n^n), \{q_j\}_{j=1}^{\infty}) = 0$ for some $p > 0$. This means that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp\left(\left\{j : e\left(h_{q_j}^{q_j}(a), h_{q_j}^{q_j}(b)\right) \geq 3p, 1 \leq j \leq n\right\}\right) = 1. \tag{13}$$

By (8),

$$e(h_n^n(a), h_n^n(b)) < e(h^n(a), h^n(b)) + 2p, \tag{14}$$

for any integer $n \geq n_p$. That is,

$$e(h^n(a), h^n(b)) > e(h_n^n(a), h_n^n(b)) - 2p > p, \tag{15}$$

for any integer $n \geq n_p$. This implies that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp(\{j : e(h^{q_j}(a), h^{q_j}(b)) \geq 3p, 1 \leq j \leq n\}) = 1, \tag{16}$$

which means that $\phi_{ab}(p, h, \{q_j\}_{j=1}^{\infty}) = 0$ for the above $p > 0$. So, D is a distributional chaotic set in the sequence $\{q_j\}_{j=1}^{\infty}$ for h.

Thus, the entire proof is completed. □

Theorem 2. *E is a distributional chaotic set for h if and only if the following conditions are true:*

(1) For any $a, b \in E : a \neq b$, $\phi_{ab}(p, (h_n^n)) = 0$ for some $p > 0$, where

$$\phi_{ab}(t, (h_n^n)) = \liminf_{n\longrightarrow\infty} \frac{1}{n} \sharp(\{1, 2, \cdots, n\} \cap \{m \in \mathbb{N} : e(h_m^m(a), h_m^m(b)) < t\}), \tag{17}$$

for any $t > 0$.

(2) For any $a, b \in E : a \neq b$, $\phi^{*}_{ab}(t, (h_n^n)) = 1$ for any $t > 0$, where

$$\phi^*_{ab}(t,(h^n_n)) = \limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp(\{1,2,\cdots,n\} \cap \{m \in \mathbb{N} : e(h^m_m(a), h^m_m(b)) < t\}), \tag{18}$$

for any $t > 0$.

Proof. Let $\{q_j\}_{j=1}^{\infty} = \{1,2,\cdots\}$, and then, by Theorem 1, the conclusion holds. □

Theorem 3. *h is DC1 if and only if there are two points $a, b \in W : a \neq b$ such that the following conditions are satisfied:*

(1) $\phi_{ab}(p,(h^n_n)) = 0$ for some $p > 0$

(2) $\phi^*_{ab}(t,(h^n_n)) = 1$ for any $t > 0$

Proof. By Theorem 2, the conclusion is obvious. So it is omitted. □

Theorem 4. *h is DC2 if and only if there are two points $a, b \in W : a \neq b$ such that the following conditions are satisfied:*

(1) $\phi_{ab}(p,(h^n_n)) < 1$ for all $p > 0$ in an interval

(2) $\phi^*_{ab}(t,(h^n_n)) = 1$ for any $t > 0$

Proof (necessity). Assume that h is DC2. By the similar proof as Theorem 1, condition (2) holds.

Let D is a distributional chaotic set of DC2 of h. Then, for any two points a and b in D, $\phi_{ab}(3p,h) < 1$ for some interval $(c,d) \subset [0,1]$ and any $p \in (c,d)$, which means that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp\left(\left\{j : e\left(h^j(a), h^j(b)\right) \geq 3p, 1 \leq j \leq n\right\}\right) > 0, \tag{19}$$

for any $p \in (c,d)$. By (4), for the above $p \in (c,d)$, there is an integer $n_p > 0$ such that (8) holds for any $a, b \in W$ and any integer $n \geq n_p$. Therefore,

$$e(h^n_n(a), h^n_n(b)) > e(f^n(a), h^n(b)) - 2p > p, \tag{20}$$

for any integer $n \geq n_p$. This implies that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp\left(\left\{j : e\left(h^j_j(a), h^j_j(b)\right) \geq 3p, 1 \leq j \leq n\right\}\right) > 0. \tag{21}$$

That is, $\phi_{ab}(p,(h^n_n)) < 1$ for any $p \in (c,d)$. Hence, condition (1) holds.

(Sufficiency) assume that conditions (1) and (2) hold. By the above discussion, we get that (12) holds for any $m \geq n_t$ and any $t > 0$.

Since $\phi^*_{ab}((1/3)t,(h^n_n)) = 1$ for any $t > 0$ and for the above a and b, then $\phi^*_{ab}(t,h) = 1$ for any $t > 0$ and for these two points a and b. By hypothesis, we obtain that for the above two points a and b, $\phi_{ab}(3p,(h^n_n)) < 1$ for some interval $(c,d) \subset [0,1]$ and any $p \in (c,d)$. This means that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp\left(\left\{j : e\left(h^j_j(a), h^j_j(b)\right) \geq 3p, 1 \leq j \leq n\right\}\right) > 0, \tag{22}$$

for any $p \in (c,d)$. By (8),

$$e(h^n(a), h^n(b)) > e(h^n_n(a), h^n_n(y)) - 2p > p, \tag{23}$$

for any integer $n \geq n_p$. This implies that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp\left(\left\{j : e\left(h^j(a), h^j(b)\right) \geq 3p, 1 \leq j \leq n\right\}\right) > 0. \tag{24}$$

That is, $\phi_{ab}(p,h) < 1$ for any $p \in (c,d)$. By the definition, h is DC2.

Thus, the entire proof is ended. □

Theorem 5. *If h is DC3, then there are $a, b \in W : a \neq b$ and some interval $(c,d) \subset [0,1]$ such that for any $p \in ((1/3)c,(1/3)d)$,*

$$\phi_{ab}(p,(h^n_n)) \leq \phi_{ab}(3p,h) < \phi^*_{ab}(3p,h) \leq \phi^*_{ab}(9p,(h^n_n)). \tag{25}$$

Proof. Suppose that h is DC3. By the definition, there is a, $b \in W : a \neq b$ such that $\phi_{ab}(t,h) < \phi^*_{ab}(t,h)$ for some interval $(c,d) \subset [0,1]$ and any $t \in (c,d)$. By the assumption that

$$\lim_{n\longrightarrow\infty} e_\infty(h^n_n, h^n) = 0, \tag{26}$$

there is some positive integer n_t such that

$$e_\infty(h^m_m, h^m) < t, \tag{27}$$

for any $m \geq n_t$. Then, we have

$$\begin{aligned} e(h^m_m(a), h^m_m(b)) &\leq e(h^m_m(a), h^m(a)) + e(h^m(a), h^m(b)) \\ &\quad + e(h^m_m(b), h^m(b)) < t + t + t = 3t, \end{aligned} \tag{28}$$

for any $m \geq n_t$. This implies that $\phi^*_{ab}(3t,(h^n_n)) \geq \phi^*_{ab}(t,h)$. Clearly, for the above two points a and b and any $p > 0$,

$$1 - \phi_{ab}(3p,h) = \limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp\left(\left\{j : e\left(h^j(a), h^j(b)\right) \geq 3p, 1 \leq j \leq n\right\}\right). \tag{29}$$

By (4), for the above $p > 0$, there is an integer $n_p > 0$ such that (8) holds for any $a, b \in W$ and any integer $n \geq n_p$. Therefore,

$$e(h^n_n(a), h^n_n(b)) > e(h^n(a), h^n(b)) - 2p > p, \tag{30}$$

for any integer $n \geq n_p$. This implies that

$$1 - \phi_{ab}(p, (h_n^n)) = \limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp\left(\left\{ j : e\left(h_j^j(a), h_j^j(y)\right) \geq p, 1 \leq j \leq n \right\}\right) \geq 1 - \phi_{ab}(3p, h). \tag{31}$$

That is, $\phi_{ab}(p, (h_n^n)) \leq \phi_{ab}(3p, h)$ for any $p > 0$. Consequently, for any $p \in ((1/3)a, (1/3)b)$,

$$\phi_{ab}(p, (h_n^n)) \leq \phi_{xy}(3p, h) < \phi_{ab}^*(3p, h) \leq \phi_{ab}^*(9p, (h_n^n)). \tag{32}$$

Thus, the proof is ended. □

Now, we will give some necessary conditions for that the limit map is DC1, DC2, or DC3. First, the following lemma which comes from [2] is needed.

Lemma 6. *For any $\varepsilon > 0$ and any integer $l > 0$, there is an integer $n_0 > 0$ (possibly depending on l) satisfying that for any integer $n > n_0$, $e(h_n^l(a), h^l(a)) < \varepsilon$ for any $a \in W$.*

Theorem 7. *Let $\{q_j\}_{j=1}^{\infty}$ be a strictly infinitely increasing sequence of positive integers. If the integer n_0 in Lemma 6 is independent of l and h is distributional chaotic in a sequence $\{q_j\}_{j=1}^{\infty}$, then there is an uncountable distributional chaotic set $E \subset W$ in the sequence $\{q_j\}_{j=1}^{\infty}$ for h such that the following conclusions are true:*

(1) For any $a, b \in E : a \neq b$ and any $t > 0$, there is an integer N_t such that $\phi_{ab}^*(t, h_n, \{q_j\}) = 1$ for any integer $n > N_t$

(2) For any $a, b \in E : a \neq b$, there are some $p > 0$ and some integer n_p such that $\phi_{ab}(p, h_n, \{q_j\}) = 0$ for any integer $n > n_p$

Proof. Suppose that h is distributional chaotic in a sequence $\{q_j\}_{j=1}^{\infty}$.

(1) By the definition, there is an uncountable distributional chaotic set $D \subset W$ in the sequence $\{q_j\}_{j=1}^{\infty}$ for h such that for any given $a, b \in D : a \neq b$, $\phi_{ab}^*((1/3)t, h, \{q_j\}_{j=1}^{\infty}) = 1$ for any $t > 0$. By hypothesis and Lemma 6, for any given $t > 0$, there is N_t, which is independent of l, such that for any $n > N_t$,

$$e\left(h_n^l(a), h^l(a)\right) < \frac{1}{3}t, \tag{33}$$

for any integer $l \geq 1$ and any $a \in W$. Therefore,

$$e\left(h_n^j(a), h_n^j(b)\right) \leq e\left(h^j(a), h^j(b)\right) + e\left(h_n^j(a), h^j(a)\right) + e\left(h_n^j(b), h^j(b)\right) < 3 \cdot \frac{1}{3}t = t, \tag{34}$$

for any integer $n > N_t$, any integer $j > 0$, and any $a, b \in D$. Consequently, $\phi_{ab}^*((1/3)t, h, \{q_j\}_{j=1}^{\infty}) = 1$ implies that $\phi_{ab}^*(t, h_n, \{q_j\}_{j=1}^{\infty}) = 1$ for any $n > N_t$ and any $t > 0$.

(2) Since h is distributional chaotic in a sequence $\{q_j\}_{j=1}^{\infty}$, then for any given $a, b \in W : a \neq b$, $\phi_{ab}(3p, h, \{q_j\}_{j=1}^{\infty}) = 0$ for some $p > 0$. This means that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \sharp(\{ j : e(h^{q_j}(a), h^{q_j}(b)) \geq 3p, 1 \leq j \leq n \}) = 1. \tag{35}$$

By hypothesis and the above argument, for the above $p > 0$, there is an integer $n_p > 0$ such that

$$\left| e\left(h_n^j(a), h_n^j(b)\right) - e\left(h^j(a), h^j(b)\right) \right| \leq e(h_n^n(a), h^n(a)) + e(h_n^n(b), h^n(b)) < p + p = 2p, \tag{36}$$

for any $a, b \in W$, any integer $n \geq n_p$, and any integer $j > 0$. Therefore,

$$e\left(h^j(a), h^j(b)\right) < e\left(h_n^j(a), h_n^j(b)\right) + 2p, \tag{37}$$

for any integer $n \geq n_p$ and any integer $j > 0$. That is,

$$e\left(h_n^j(a), h_n^j(b)\right) > e\left(h^j(a), h^j(b)\right) - 2p > p, \tag{38}$$

for any integer $n \geq n_p$. This implies that

$$\limsup_{m\longrightarrow\infty} \frac{1}{m} \sharp\left(\left\{ j : e\left(h_n^{q_j}(a), h_n^{q_j}(b)\right) \geq 3p, 1 \leq j \leq m \right\}\right) = 1, \tag{39}$$

for any integer $n \geq n_p$. That is, $\phi_{ab}(p, h_n, \{q_j\}_{j=1}^{\infty}) = 0$ for the above $p > 0$ and any integer $n \geq n_p$.

Thus, the proof is finished. □

Theorem 8. *If the integer n_0 in Lemma 6 is independent of l and h is DC1, then there is a distributional chaotic set $E \subset W$ for h which is uncountable such that the following hold:*

(1) For any $a, b \in E : a \neq b$ and any $t > 0$, there is an integer N_t such that $\phi_{ab}^*(t, h_n) = 1$ for any integer $n > N_t$

(2) For any $a, b \in E : a \neq b$, there are some $p > 0$ and some integer n_p such that $\phi_{ab}(p, h_n) = 0$ for any integer $n > n_p$

Proof. Since DC1 is the special case of distributional chaotic in a sequence, then it is obvious. □

Theorem 9. *If the integer n_0 in Lemma 6 is independent of l and h is DC2, then there are two points $a, b \in W$ with $a \neq b$ such that the following are true:*

(1) *For any $t > 0$, there is an integer N_t such that $\phi^*_{ab}(t, h_n) = 1$ for any integer $n > N_t$*

(2) *There are some interval $(c, d) \subset [0, 1]$ such that for any $p > 0$, there is some integer n_p such that $\phi_{ab}(p, h_n) < 1$ for any integer $n > n_p$*

Proof. Suppose that h is DC2.

(1) The proof is similar to Theorem 7. So it is omitted

(2) By the definition, for any given $a, b \in D : a \neq b$, $\phi_{ab}(3p, h) < 1$ for some interval $(c, d) \subset [0, 1]$ and any $p \in (c, d)$. This means that for any $p \in (c, d)$,

$$\limsup_{n \longrightarrow \infty} \frac{1}{n} \sharp \left(\left\{ j : e\left(h^j(a), h^j(b)\right) \geq 3p, 1 \leq j \leq n \right\} \right) > 0. \quad (40)$$

By hypothesis and the above argument, for any given $p \in (c, d)$, there is an integer $n_p > 0$ such that (36) holds for any $a, b \in W$, any integer $n \geq n_p$, and any integer $j > 0$. Therefore,

$$e\left(h_n^j(a), h_n^j(b)\right) > e\left(h^j(a), h^j(b)\right) - 2p > p, \quad (41)$$

for any integer $n \geq n_p$ and any integer $j > 0$. This implies that

$$\limsup_{m \longrightarrow \infty} \frac{1}{m} \sharp \left(\left\{ j : e\left(h_n^j(a), h_n^j(b)\right) \geq 3p, 1 \leq j \leq m \right\} \right) > 0, \quad (42)$$

for any integer $n \geq n_p$. That is, $\phi_{ab}(p, h_n) < 1$ for any $p \in (c, d)$ and any integer $n \geq n_p$.

Thus, the proof is complete. □

Theorem 10. *If the integer n_0 in Lemma 6 is independent of l and h is DC3, then there are $a, b \in W : a \neq b$ and some interval $(c, d) \subset [0, 1]$, for any $p \in ((1/3)c, (1/3)d)$, there is an integer $N_p > 0$ such that*

$$\phi_{ab}(p, h_n) \leq \phi_{ab}(3p, h) < \phi^*_{ab}(3p, h) \leq \phi^*_{ab}(9p, h_n), \quad (43)$$

for any $n > N_p$.

Proof. Suppose that h is DC3. Then, by the definition, there are two points $a, b \in W : a \neq b$ such that $\phi_{ab}((1/3)t, h) < \phi^*_{ab}((1/3)t, h)$ for some interval $(c, d) \subset [0, 1]$ and any $t \in (c, d)$. By hypothesis and Lemma 6, for any given $t > 0$, there is N_t, which is independent of l, such that for any $n > N_t$,

$$e\left(h_n^l(a), h^l(a)\right) < \frac{1}{3}t, \quad (44)$$

for any integer $l \geq 1$ and any $a \in W$. So,

$$e\left(h_n^j(a), h_n^j(b)\right) \leq e\left(h^j(a), h^j(b)\right) + e\left(h_n^j(a), h^j(a)\right) + e\left(h_n^j(b), h^j(b)\right) < 3 \cdot \frac{1}{3}t = t, \quad (45)$$

for any integer $n > N_t$, any integer $j > 0$, and any $a, b \in D$. This implies that

$$\phi^*_{ab}(t, h_n) \geq \phi^*_{ab}\left(\frac{1}{3}t, h\right), \quad (46)$$

for any $n > N_t$ and any $t \in (c, d)$. By the definition, for any $p > 0$ and the above two points a and b,

$$1 - \phi_{ab}(3p, h) = \limsup_{n \longrightarrow \infty} \frac{1}{n} \sharp \left(\left\{ j : e\left(h^j(a), h^j(b)\right) \geq 3p, 1 \leq j \leq n \right\} \right). \quad (47)$$

By hypothesis and the above argument, for the above $p > 0$, there is an integer $n_p > 0$ such that (36) holds for any $a, b \in W$, any integer $n \geq n_p$, and any integer $j > 0$. Therefore,

$$e\left(h_n^j(a), h_n^j(b)\right) > e\left(h^j(a), h^j(b)\right) - 2p > p, \quad (48)$$

for any integer $n \geq n_p$ and any integer $j > 0$. This implies that

$$\begin{aligned} 1 - \phi_{ab}(p, h_n) &= \limsup_{m \longrightarrow \infty} \frac{1}{m} \sharp \left(\left\{ j : e\left(h_n^j(a), h_n^j(b)\right) \geq p, 1 \leq j \leq m \right\} \right) \\ &\geq \limsup_{n \longrightarrow \infty} \frac{1}{n} \sharp \left(\left\{ j : e\left(h^j(a), h^j(b)\right) \geq 3p, 1 \leq j \leq n \right\} \right) \\ &= 1 - \phi_{ab}(3p, h), \end{aligned} \quad (49)$$

for any integer $n \geq n_p$ and any $p > 0$. That is, $\phi_{ab}(p, h_n) \leq \phi_{ab}(3p, h)$ for the above $p > 0$ and any integer $n \geq n_p$.

Thus, the proof is ended. □

Remark 11. There are some other future studies; for example, the following conclusion is true or not.

Under the notations and the assumptions of Theorem 10, there exists an integer $N > 0$ such that h_n is DC3 for any $n \geq N$.

Data Availability

The data used to support the findings of this study are currently under embargo. Requests for data, 6 months after publication of this article, will be considered by the corresponding author.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

Risong Li and Tianxiu Lu were responsible for conceptualization; Risong Li, Tianxiu Lu, and Jingmin Pi were responsible for validation; Tianxiu Lu and Waseem Anwar were responsible for formal analysis; Jingmin Pi was responsible for investigation; Risong Li was responsible for writing original draft preparation; Tianxiu Lu and Waseem Anwar were responsible for writing, review, and editing; Tianxiu Lu was responsible for supervision; Risong Li and Tianxiu Lu were responsible for funding acquisition.

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