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Research Article

An Application of the t-Extension of the p-Fibonacci Pascal Matrix in Coding Theory

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We consider the t-extension of the p-Fibonacci Pascal matrix. First, we study the k-th power of the t-extension of the p-Fibonacci lower and upper triangular Pascal matrix. Then, we obtain a new code which is named the t-extension of the p-Fibonacci Pascal matrix coding/decoding by using them.

1. Introduction

The Fibonacci sequence is defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}, n \geq 3$, with the initial values $f_1 = f_2 = 1$. Recently, there are many papers devoted to the study of the Fibonacci and generalized Fibonacci sequences; for example, see [1, 2]. Such the generalization, which will be used in this paper, is the t-extension of the p-Fibonacci sequence.

Definition 1. For constants $t \geq 1$ and $p \geq 0$, the t-extension of the p-Fibonacci sequence $\{f^p(t,n)\}_{n=0}^{\infty}$ is given with the following recurrence relation (see [3, 4]):

$$f^p(t,n) = \begin{cases} 0, & n < 1, \\ 1, & n = 1, \\ tf^p(t,n-1) + f^p(t,n-p-1), & n > 1. \end{cases} \quad (1)$$

For example, for $t = 3$ and $p = 2$, we have $f^2(3,n) = 3f^2(3,n-1) + f^2(3,n-3)$ and

$$\left\{f^2(3,n)\right\}_{-\infty}^{\infty} = \{\cdots,0,1,3,9\cdots\}. \quad (2)$$

In [4–13], the authors studied the properties of the Pascal matrix. We recall the following definition from [5].

Definition 2. The $n \times n$ lower triangular Pascal matrix $L_n = [l_{ij}]$ is defined as

$$l_{i,j} = \begin{cases} \begin{pmatrix} i \\ j \end{pmatrix}, & \text{if } i \geq j, \\ 0, & \text{otherwise}, \end{cases} \quad (3)$$

and the $n \times n$ upper triangular matrix $U_n = [u_{ij}]$ is defined as

$$u_{i,j} = \begin{cases} \begin{pmatrix} i \\ j \end{pmatrix}, & \text{if } j \geq i, \\ 0, & \text{otherwise}. \end{cases} \quad (4)$$

The $n \times n$, t-extension of the p-Fibonacci lower triangular Pascal matrix, denoted by $L^p(t,n)$, is defined as follows:

$$L^p(t,n) = \left[f_{i,j}\right]_{i,j=1,\cdots,n} = \begin{cases} f^p(t,i-j+1), & \text{if } i-j+1 \geq 0, \\ 0, & \text{if } i-j+1 < 0, \end{cases} \quad (5)$$

and the inverse of the t-extension of the p-Fibonacci lower triangular Pascal matrix, $(L^p(t,n))^{-1} = [l'_{i,j}]_{i,j=1,\cdots,n}$, is given as

$$l'_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ -t, & \text{if } j = i-1, \\ -1, & \text{if } j = i-(p+1), \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

See [14] for more details.

For example, we have

$$L^1(3,4) = \begin{bmatrix} f_{3,1} & 0 & 0 & 0 \\ f_{3,2} & f_{3,1} & 0 & 0 \\ f_{3,3} & f_{3,2} & f_{3,1} & 0 \\ f_{3,4} & f_{3,3} & f_{3,2} & f_{3,1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 10 & 3 & 1 & 0 \\ 33 & 10 & 3 & 1 \end{bmatrix},$$

$$L^1(3,4)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ -1 & -t & 1 & 0 \\ 0 & -1 & -t & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{bmatrix}. \tag{7}$$

Now, we define the t-extension of the p-Fibonacci upper triangular Pascal matrix, denoted by $U^p(t,n)$, as follows:

$$U^p(t,n) = [m_{i,j}]_{i,j=1,\cdots,n} = \begin{cases} f^p(t, j-i+1), & \text{if } j-i+1 \geq 0, \\ 0, & \text{if } j-i+1 < 0. \end{cases} \tag{8}$$

The inverse of the t-extension of the p-Fibonacci upper triangular Pascal matrix denoted by $U^p(t,n)^{-1}$ is defined as follows:

$$U^p(t,n)^{-1} = \left[m'_{i,j}\right]_{i,j=1,\cdots,n} = \begin{cases} 1, & \text{if } i = j, \\ -t, & \text{if } i = j-1, \\ -1, & \text{if } i = j-(p+1), \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

An application of the Fibonacci matrix and its generalization is in coding theory (see [15–22]). In this paper, using the t-extension of the p-Fibonacci lower and upper triangular Pascal matrix, we present a new code matrix.

The rest of the paper is organized as follows: in Section 2, we obtain the k-th power of the t-extension of the p-Fibonacci lower triangular matrix and upper triangular Pascal matrix. Also, we give the inverse of them $(2 \leq n \leq 4)$. Section 3 is devoted to obtaining a coding and decoding on the t-extension of the p-Fibonacci Pascal matrix.

2. The k-th Power of the t-Extension of the p-Fibonacci

In this section, we calculate the k-th power of the t-extension of the p-Fibonacci lower triangular Pascal matrix, the t-extension of the p-Fibonacci upper triangular Pascal matrix, and their inverse matrices. First, we denote the k-th power of $L^p(t,n)$ and $U^p(t,n)$ by $(L^p(t,n))^k$ and $(U^p(t,n))^k$, respectively.

Theorem 3. *For $p = 1$ and $t, k \geq 1$, we have*

$$(i) L^1(t,2)^k = \begin{bmatrix} 1 & 0 \\ kt & 1 \end{bmatrix}, \quad L^1(t,3)^k = \begin{bmatrix} 1 & 0 & 0 \\ kt & 1 & 0 \\ \binom{k}{2} t^2 + k(t^2+1) & kt & 1 \end{bmatrix},$$

$$\left(L^1(t,4)\right)^k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ kt & 1 & 0 & 0 \\ \binom{k}{2} t^2 + k(t^2+1) & kt & 1 & 0 \\ b_k(t^3+t) + k(t^3+2t) + \sum_{i=2}^{k} \left(\binom{i}{2} t^2 + i(t^2+1) \right) t & \binom{k}{2} t^2 + k(t^2+1) & kt & 1 \end{bmatrix}, \tag{10}$$

where $b_1 = 0$, $b_2 = 2$, and $b_k = b_{k-1} + (k-1)$ for $k \geq 3$.

$$(ii) U^1(t,2)^k = \begin{bmatrix} 1 & kt \\ 0 & 1 \end{bmatrix}, \quad U^1(t,3)^k = \begin{bmatrix} 1 & kt & \binom{k}{2}t^2 + k(t^2+1) \\ 0 & 1 & kt \\ 0 & 0 & 1 \end{bmatrix},$$

$$\left(U^1(t,4)\right)^k = \begin{bmatrix} 1 & tk & \binom{k}{2}t^2 + k(t^2+1) & b_k(t^3+t) + k(t^3+2t) + \sum_{i=2}^{k}\left(\binom{i}{2}t^2 + i(t^2+1)\right)t \\ 0 & 1 & tk & \binom{k}{2}t^2 + k(t^2+1) \\ 0 & 0 & 1 & tk \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

where $b_1 = 0$, $b_2 = 2$, and $b_k = b_{k-1} + (k-1)$, $k \geq 3$.

Proof.

(i) Since the proof of the result for $n = 2$, 3, and 4 is similar, we prove the result by induction on k for $n = 4$. Let $k = 1$. Then, by relation (3), we have

$$L^1(t,4) = \begin{bmatrix} f^1(t,1) & 0 & 0 & 0 \\ f^1(t,2) & & 0 & 0 \\ f^1(t,3) & f^1(t,2) & f^1(t,1) & 0 \\ f^1(t,4) & f^1(t,3) & f^1(t,2) & f^1(t,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2+1 & t & 1 & 0 \\ t^3+2t & t^2+1 & t & 1 \end{bmatrix}. \tag{12}$$

Suppose the result holds for $k = l$. Therefore, we have

$$\begin{aligned} &\left(L^1(t,n)\right)^{l+1} \\ &= \left(L^1(t,n)\right)^l \times \left(L^1(t,n)\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ lt & 1 & 0 & 0 \\ \binom{l}{2}t^2 + l(t^2+1) & lt & 1 & 0 \\ b_l(t^3+t) + l(t^3+2t) + \sum_{i=2}^{l}\left(\binom{i}{2}t^2 + i(t^2+1)\right)t & \binom{l}{2}t^2 + l(t^2+1) & lt & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2+1 & t & 1 & 0 \\ t^3+2t & t^2+1 & t & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ (l+1)t & 1 & 0 & 0 \\ \binom{l+1}{2}t^2 + (l+1)(t^2+1) & lt & 1 & 0 \\ u & \binom{l+1}{2}t^2 + (l+1)(t^2+1) & (l+1)t & 1 \end{bmatrix}, \end{aligned} \tag{13}$$

where

$$\begin{aligned} u = b_{l+1}(t^3+t) + l(t^3+2t) \\ + \sum_{i=2}^{l}\left(\binom{i}{2}t^2 + i(t^2+1)\right)t\binom{l+1}{2}t^2 \\ + (l+1)(t^2+1). \end{aligned} \tag{14}$$

(ii) The proof is similar to the proof of the proposition (i) and then is omitted

□

Example 1. For $n = 4$ and $k = 3$, we have

$$\left(L^1(t,4)\right)^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3t & 1 & 0 & 0 \\ 3t^2 + 3(t^2+1) & 3t & 1 & 0 \\ 4(t^3+t) + 3(t^3+2t) + (t^2 + 2(t^2+1))t & 3t^2 + 3(t^2+1) & 3t & 1 \end{bmatrix}. \tag{15}$$

Similar to the proof of Theorem 3, we can prove the following theorems.

Theorem 4. *For $n = 4$, $p = 2$, and $t \geq 1$, we have*

$$(i)\left(L^2(t,n)\right)^k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ kt & 1 & 0 & 0 \\ \left(\binom{k}{2}+k\right)t^2 & kt & 1 & 0 \\ \left(4+\sum_{i=3}^{k}\left(\binom{i-1}{2}+2i-1\right)\right)t^3+k & \left(\binom{k}{2}+k\right)t^2+k\left(t^2+1\right) & kt & 1 \end{bmatrix},$$

$$(ii)\left(U^2(t,n)\right)^k = \begin{bmatrix} 1 & tk & \left(\binom{k}{2}+k\right)t^2 & \left(4+\sum_{i=3}^{k}\left(\binom{i-1}{2}+2i-1\right)\right)t^3+k \\ 0 & 1 & tk & \left(\binom{k}{2}+k\right)t^2 \\ 0 & 0 & 1 & tk \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (16)$$

Theorem 5. *For $n = 4$, $p \geq 3$, and $t \geq 1$, we have*

$$(i)\left(L^p(t,n)\right)^k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ kt & 1 & 0 & 0 \\ \left(\binom{k}{2}+k\right)t^2 & kt & 1 & 0 \\ \left(4+\sum_{i=3}^{k}\left(\binom{i-1}{2}+2i-1\right)\right)t^3 & \left(\binom{k}{2}+k\right)t^2+k\left(t^2+1\right) & kt & 1 \end{bmatrix},$$

$$(ii)\left(U^p(t,n)\right)^k = \begin{bmatrix} 1 & tk & \left(\binom{k}{2}+k\right)t^2 & \left(4+\sum_{i=3}^{k}\left(\binom{i-1}{2}+2i-1\right)\right)t^3 \\ 0 & 1 & tk & \left(\binom{k}{2}+k\right)t^2 \\ 0 & 0 & 1 & tk \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (17)$$

We denote the inverse of $(L^p(t,n))^k$ and $(U^p(t,n))^k$ by $(L^p(t,n))^{-k}$ and $(U^p(t,n))^{-k}$, respectively. Now, we have the following theorem.

Theorem 6. *For $n=4$ and $p=1$,*

$$(i)L^1(t,2)^{-k}=\begin{bmatrix}1&0\\-kt&1\end{bmatrix},\quad L^1(t,3)^{-k}=\begin{bmatrix}1&0&0\\-kt&1&0\\\binom{k}{2}t^2-k&-kt&1\end{bmatrix},$$

$$\left(L^1(t,4)\right)^{-k}=\begin{bmatrix}1&0&0&0\\-tk&1&0&0\\\binom{k}{2}t^2-k&-tk&1&0\\c_k&\binom{k}{2}t^2-k&-tk&1\end{bmatrix},\tag{18}$$

where $c_1=0$, $c_2=2t$, and

$$c_k=c_{k-1}+(k-1)t-\sum_{i=3}^{k}\left(\binom{i-1}{2}t^2-(i-1)\right)t,\tag{19}$$

for $k\geq 3$.

$$(ii)U^1(t,2)^{-k}=\begin{bmatrix}1&-kt\\0&1\end{bmatrix},\quad U^1(t,3)^{-k}=\begin{bmatrix}1&-tk&\binom{k}{2}t^2-k\\0&1&-tk\\0&0&1\end{bmatrix},$$

$$\left(U^1(t,n)\right)^{-k}=\begin{bmatrix}1&-tk&\binom{k}{2}t^2-k&c_k\\0&1&-tk&\binom{k}{2}t^2-k\\0&0&1&-tk\\0&0&0&1\end{bmatrix},\tag{20}$$

where $c_1=0$, $c_2=2t$, and

$$c_k=c_{k-1}+(k-1)t-\sum_{i=3}^{k}\left(\binom{i-1}{2}t^2-(i-1)\right)t,\ k\geq 3.\tag{21}$$

Proof.

(i) It is sufficient to show that the result holds for $n=4$. For this, we use the induction method on k. Let $k=1$. Then, by relation (4), we have

$$\left(L^1(t,n)\right)^{-k}=\begin{bmatrix}1&0&0&0\\-t&1&0&0\\-1&-t&1&0\\0&-1&-t&1\end{bmatrix}.\tag{22}$$

Suppose the statement holds for $k=l$. Then, we have

$$\begin{aligned}&\left(L^1(t,n)\right)^{-(l+1)}\\&=\left(L^1(t,n)\right)^{-l}\times\left(L^1(t,n)\right)^{-1}\\&=\begin{bmatrix}1&0&0&0\\-lt&1&0&0\\\binom{l}{2}t^2-l&-lt&1&0\\c_l&\binom{l}{2}t^2-l&-lt&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0\\-t&1&0&0\\-1&-t&1&0\\0&-1&-t&1\end{bmatrix}\\&=\begin{bmatrix}1&0&0&0\\-(l+1)t&1&0&0\\\binom{l+1}{2}t^2-(l+1)&-(l+1)t&1&0\\c_l+\left(\binom{l}{2}t^2-l\right)t+lt&\binom{l+1}{2}t^2-l&-(l+1)t&1\end{bmatrix}.\end{aligned}\tag{23}$$

Since

$$\begin{aligned}c_l-\left(\binom{l}{2}t^2-l\right)t+lt&=c_l+lt\\&\quad-\sum_{i=3}^{l}\left(\binom{i-1}{2}t^2-(i-1)\right)t\\&=c_{l+1},\end{aligned}\tag{24}$$

we have result.

The proof (ii) is similar to the proof of the proposition (i). So, we omit it. □

Example 2. For $n = 4$ and $t = 5$, we have

$$(L^1(t,n))^{-5} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5t & 1 & 0 & 0 \\ -10t^2 - 5 & -5t & 1 & 0 \\ 11t - (t^2 - 2)t - (3t^2 - 3)t - (6t^2 - 4)t & -10t^2 - 5 & -5t & 1 \end{bmatrix}. \tag{25}$$

By induction on k, we can prove the following theorems and therefore we will omit their proofs.

Theorem 7. *For $n = 4$ and $p = 2$,*

$$(i)(L^p(t,4))^{-k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -tk & 1 & 0 & 0 \\ \binom{k}{2}t^2 & -tk & 1 & 0 \\ -\sum_{i=3}^{k}\binom{i-1}{2}t^3 - k & \binom{k}{2}t^2 & -tk & 1 \end{bmatrix},$$

$$(ii)(U^p(t,n))^{-k} = \begin{bmatrix} 1 & -tk & \binom{k}{2}t^2 & -\sum_{i=3}^{k}\binom{i-1}{2}t^3 - k \\ 0 & 1 & -tk & \binom{k}{2}t^2 \\ 0 & 0 & 1 & -tk \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{26}$$

Theorem 8. *For $n = 4$ and $p \geq 3$,*

$$(i)(L^p(t,n))^{-k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -tk & 1 & 0 & 0 \\ \binom{k}{2}t^2 & -tk & 1 & 0 \\ -\sum_{i=3}^{k}\binom{i-1}{2}t^3 & \binom{k}{2}t^2 & -tk & 1 \end{bmatrix},$$

$$(ii)(U^p(t,n))^{-k} = \begin{bmatrix} 1 & -tk & \binom{k}{2}t^2 & -\sum_{i=3}^{k}\binom{i-1}{2}t^3 \\ 0 & 1 & -tk & \binom{k}{2}t^2 \\ 0 & 0 & 1 & -tk \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{27}$$

Lemma 9. *For $m, s, p \in \mathbb{N}$ and $n = 4$, we get*

$$\begin{aligned}(i)\left(L^P(t,n)\right)^m \times \left(L^P(t,n)\right)^s &= \left(L^P(t,n)\right)^s \times \left(L^P(t,n)\right)^m \\ &= \left(L^P(t,n)\right)^{m+s}, \\ (ii)\left(U^P(t,n)\right)^m \times \left(U^P(t,n)\right)^s &= \left(U^P(t,n)\right)^s \times \left(U^P(t,n)\right)^m \\ &= \left(U^P(t,n)\right)^{m+s}.\end{aligned} \tag{28}$$

Proof.

(i) By Theorem 3 for $p = 1$, we have

$$\begin{aligned}\left(L^1(t,n)\right)^m \times \left(L^1(t,n)\right)^s &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ mt & 1 & 0 & 0 \\ \binom{m}{2}t^2 + m(t^2+1) & mt & 1 & 0 \\ b_m(t^3+t) + m(t^3+2t) + \sum_{i=2}^{m}\left(\binom{i}{2}t^2 + i(t^2+1)\right)t & \binom{m}{2}t^2 + m(t^2+1) & mt & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ st & 1 & 0 & 0 \\ \binom{s}{2}t^2 + s(t^2+1) & st & 1 & 0 \\ b_s(t^3+t) + s(t^3+2t) + \sum_{i=2}^{s}\left(\binom{i}{2}t^2 + i(t^2+1)\right)t & \binom{s}{2}t^2 + s(t^2+1) & kt & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ (m+s)t & 1 & 0 & 0 \\ \binom{m+s}{2}t^2 + (m+s)(t^2+1) & (m+s)t & 1 & 0 \\ a & b & (m+s)t & 1 \end{bmatrix} = \left(L^1(t,n)\right)^{m+s},\end{aligned} \tag{29}$$

where

$$a = b_{m+s}\left(t^3+t\right) + (m+s)\left(t^3+2t\right) + \sum_{i=2}^{m+s}\left(\binom{i}{2}t^2 + i\left(t^2+1\right)\right)t,$$
$$b = \binom{m+s}{2}t^2 + (m+s)\left(t^2+1\right). \quad (30)$$

In a similar way, one can prove the remaining cases. □

3. Coding and Decoding on k-th Power the t-Extension of the p-Fibonacci Pascal Matrix

In the present section, first, we introduce a coding and decoding on the t-extension of the p-Fibonacci Pascal matrix and get its error detection and correction.

For $2 \le n \le 4$ and an initial message $M_{n\times n}$, we will name a transformation $E = (L^p(t,n))^k \times M \times (U^p(t,n))^k$ as the t-extension of the p-Fibonacci Pascal coding and a transformation $M = (L^p(t,n))^{-k} \times E \times (U^p(t,n))^{-k}$ as the t-extension of the p-Fibonacci Pascal decoding. Note that the matrix E is as a code matrix and all of the elements of M are positive.

Lemma 10. *For $2 \le n \le 4$ and $k, p \in \mathbb{N}$,* $\det(E) = \det(M)$.

Proof. Since $E = (L^p(t,n))^k \times M \times (U^p(t,n))^k$, we have

$$\begin{aligned}\det(E) &= \det\left((L^p(t,n))^k \times M \times U^p(t,n)\right)^k\Big) \\ &= \det\left(U^p(t,n))^k\right) \times \det(M) \times \det\left(L^p(t,n))^k\right) \\ &= 1 \times \det(M) \times 1 = \det(M).\end{aligned} \quad (31)$$

□

Now, we obtain a relation for elements of the code matrix E. We study three cases. The rest cases are similar.

Case 1. Let $n = 2$ and $p = 1$. For the matrix $M_{2\times 2}$, we have

$$\begin{aligned}E &= \left(L^1(t,2)\right)^k \times M \times \left(U^1(t,2)\right)^k \\ &= \begin{bmatrix}1 & 0\\ tk & 1\end{bmatrix}\begin{bmatrix}m_1 & m_2\\ m_3 & m_4\end{bmatrix}\begin{bmatrix}1 & tk\\ 0 & 1\end{bmatrix} \\ &= \begin{bmatrix}m_1 & tkm_1+m_2\\ tkm_1+m_3 & t^2k^2m_1 + tk(m_3+m_2)+m_4\end{bmatrix}.\end{aligned} \quad (32)$$

On the other hand, we get

$$\begin{aligned}M &= \left(L^1(t,2)\right)^{-k} E \times \left(U^1(t,2)\right)^{-k} \\ &= \begin{bmatrix}1 & 0\\ -tk & 1\end{bmatrix}\begin{bmatrix}e_1 & e_2\\ e_3 & e_4\end{bmatrix}\begin{bmatrix}1 & -tk\\ 0 & 1\end{bmatrix} \\ &= \begin{bmatrix}e_1 & -tke_1+e_2\\ -tke_1+e_3 & t^2k^2e_1 - tk(e_2+e_3)+e_4\end{bmatrix} \\ &= \begin{bmatrix}m_1 & m_2\\ m_3 & m_4\end{bmatrix}.\end{aligned} \quad (33)$$

We obtain

$$\begin{aligned}m_1 &= e_1 \ge 0,\\ m_2 &= -tke_1 + e_2 \ge 0,\\ m_3 &= -tke_1 + e_3 \ge 0,\\ m_4 &= t^2k^2e_1 - tk(e_2+e_3) + e_4 \ge 0.\end{aligned} \quad (34)$$

So, we have

$$\frac{e_1}{e_2} \le \frac{1}{tk}, \quad \frac{e_1}{e_3} \le \frac{1}{tk}, \quad \frac{e_1}{e_4} \le \frac{1}{t^2k^2}. \quad (35)$$

Case 2. Suppose that $p = 1$ and $n = 3$. For $M_{3\times 3}$,

$$\begin{aligned}M &= \left(L^1(t,3)\right)^{-k} \times E \times \left(L^1(t,3)\right)^{-k} \\ &= \begin{bmatrix}1 & 0 & 0\\ -tk & 1 & 0\\ \binom{k}{2}t^2 - k & -tk & 1\end{bmatrix}\begin{bmatrix}e_1 & e_2 & e_3\\ e_4 & e_5 & e_6\\ e_7 & e_8 & e_9\end{bmatrix} \cdot \begin{bmatrix}1 & -tk & \binom{k}{2}t^2 - k\\ 0 & 1 & -tk\\ 0 & 0 & 1\end{bmatrix} \\ &= \begin{bmatrix}e_1 & -tke_1 + e_2 & a\\ -tke_1 + e_4 & (-tke_1+e_4)(-tk) + (-tke_2+e_5) & b\\ \left(\binom{k}{2}t^2 - k\right)e_1 - tke_4 + e_7 & c & d\end{bmatrix},\end{aligned} \quad (36)$$

where

$$a = \left(\binom{k}{2}t^2 - k\right)e_1 - tke_2 + e_3,$$

$$
\begin{aligned}
b &= \left(\binom{k}{2}t^2 - k\right)(-tke_1 + e_4) + (-tke_2 + e_5)(-tk) \\
&\quad + (-tke_3 + e_6), \\
c &= \left(\left(\binom{k}{2}t^2 - k\right)e_1 - tke_4 + e_7\right)(-tk) \\
&\quad + \left(\binom{k}{2}t^2 - k\right)e_2 - tke_5 + e_8, \\
d &= \left(\left(\binom{k}{2}t^2 - k\right)e_1 - tke_4 + e_7\right)\left(\binom{k}{2}t^2 - k\right) \\
&\quad + \left(\left(\binom{k}{2}t^2 - k\right)e_2 - tke_5 + e_8\right)(-tk) \\
&\quad + \left(\binom{k}{2}t^2 - k\right)e_3 - tke_6 + e_9.
\end{aligned} \tag{37}
$$

So,

$$
\begin{aligned}
m_1 &= e_1 \geq 0, \\
m_2 &= -tke_1 + e_2 \geq 0, \\
m_3 &= \left(\binom{k}{2}t^2 - k\right)e_1 - tke_2 + e_3 \geq 0, \\
m_4 &= -tke_1 + e_4 \geq 0, \\
m_5 &= (-tke_1 + e_4)(-tk) + (-tke_2 + e_5) \geq 0, \\
m_6 &= \left(\binom{k}{2}t^2 - k\right)(-tke_1 + e_4) + (-tke_2 + e_5)(-tk) \\
&\quad + (-tke_3 + e_6) \geq 0, \\
m_7 &= \left(\binom{k}{2}t^2 - k\right)e_1 - tke_4 + e_7 \geq 0, \\
m_8 &= \left(\left(\binom{k}{2}t^2 - k\right)e_1 - tke_4 + e_7\right)(-tk) \\
&\quad + \left(\binom{k}{2}t^2 - k\right)e_2 - tke_5 + e_8 \geq 0, \\
m_9 &= \left(\left(\binom{k}{2}t^2 - k\right)e_1 - tke_4 + e_7\right)\left(\binom{k}{2}t^2 - k\right) \\
&\quad + \left(\left(\binom{k}{2}t^2 - k\right)e_2 - tke_5 + e_8\right)(-tk) \geq 0.
\end{aligned} \tag{38}
$$

Therefore, we get

$$
\frac{e_1}{e_2} \leq \frac{1}{tk}, \quad \frac{e_1}{e_3} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2 - k\right)}, \tag{39}
$$

$$
\frac{e_1}{e_4} \leq \frac{1}{tk}, \quad \frac{e_1}{e_5} \leq \frac{1}{t^2k^2}, \tag{40}
$$

$$
\frac{e_1}{e_6} \leq \frac{1}{t^3k^3 - \left(\binom{k}{2}kt^2 - k^2t\right)}, \quad \frac{e_1}{e_7} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2 - k\right)}, \tag{41}
$$

$$
\frac{e_1}{e_8} \leq \frac{1}{t^3k^3 - \left(\binom{k}{2}kt^2 - k^2t\right)}, \quad \frac{e_1}{e_9} \leq \frac{1}{\left(t^2k^2 + k - \left(\binom{k}{2}t^2\right)^2\right)}. \tag{42}
$$

Case 3. Let $p = 2$ and $n = 4$, we obtain

$$
\begin{aligned}
M &= \left(L^2(t,4)\right)^{-k} \times E \times \left(L^p(t,4)\right)^{-k} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -tk & 1 & 0 & 0 \\ \binom{k}{2}t^2 & -tk & 1 & 0 \\ -\sum_{i=3}^{k}\binom{k}{2}t^3 - k & \binom{k}{2}t^2 & -tk & 1 \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ e_5 & e_6 & e_7 & e_8 \\ e_9 & e_{10} & e_{11} & e_{12} \\ e_{13} & e_{14} & e_{15} & e_{16} \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} 1 & -tk & \binom{k}{2}t^2 & -\sum_{i=3}^{k}\binom{k}{2}t^3 - k \\ 0 & 1 & -tk & \binom{k}{2}t^2 \\ 0 & 0 & 1 & -tk \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} e_1 & -tke_1 + e_2 & \binom{k}{2}t^2e_1 - tke_2 + e_3 & h_1 \\ -tke_1 + e_5 & (-tke_1 + e_5)(-tk) + (-tke_2 + e_6) & h_2 & h_3 \\ h_4 & h_5 & h_6 & h_7 \\ h_8 & h_9 & h_{10} & h_{11} \end{bmatrix},
\end{aligned} \tag{43}
$$

where

$$
\begin{aligned}
h_1 &= \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_1 + \Bigg(\binom{k}{2}t^2\Bigg)e_2 - tke_3 + e_4,\\
h_2 &= (-tke_1 + e_5)\Bigg(\binom{k}{2}t^2\Bigg) + (-tke_2 + e_6)(-tk) - tke_3 + e_7,\\
h_3 &= (-tke_1 + e_5)\Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg) + (-tke_2 + e_6)\Bigg(\binom{k}{2}t^2\Bigg)\\
&\quad + (-tke_3 + e_7)(-tk) + (-tke_4 + e_8),\\
h_4 &= \binom{k}{2}t^2e_1 - tke_5 + e_9,\\
h_5 &= \Bigg(\binom{k}{2}t^2e_1 - tke_5 + e_9\Bigg)(-tk) + \binom{k}{2}t^2e_2 - tke_6 + e_{10},\\
h_6 &= \Bigg(\binom{k}{2}t^2e_1 - tke_5 + e_9\Bigg)\Bigg(\binom{k}{2}t^2\Bigg)\\
&\quad + \Bigg(\binom{k}{2}t^2e_2 - tke_6 + e_{10}\Bigg)(-tk) + \binom{k}{2}t^2e_3 - tke_7 + e_{11},\\
h_7 &= \Bigg(\binom{k}{2}t^2e_1 - tke_5 + e_9\Bigg)\Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)\\
&\quad + \Bigg(\binom{k}{2}t^2e_2 - tke_6 + e_{10}\Bigg)\binom{k}{2}t^2\\
&\quad + \Bigg(\binom{k}{2}t^2e_3 - tke_7 + e_{11}\Bigg)(-tk) + \binom{k}{2}t^2e_4 - tke_8 + e_{12},\\
h_8 &= \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_1 + \binom{k}{2}t^2e_5 - tke_9 + e_{13},\\
h_9 &= \Bigg(\Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_1 + \binom{k}{2}t^2e_5 - tke_9 + e_{13}\Bigg)(-tk)\\
&\quad + \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_2 + \binom{k}{2}t^2e_6 - tke_{10} + e_{14},\\
h_{10} &= \Bigg(\Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_1 + \binom{k}{2}t^2e_5 - tke_9 + e_{13}\Bigg)\binom{k}{2}t^2\\
&\quad + \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_2 + \binom{k}{2}t^2e_6 - tke_{10} + e_{14}\Bigg)(-tk)\\
&\quad + \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_3 + \binom{k}{2}t^2e_7 - tke_{11} + e_{15},\\
h_{11} &= \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_1 + \binom{k}{2}t^2e_5 - tke_9 + e_{13}\Bigg)\\
&\quad \cdot \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg) + \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_2\\
&\quad + \binom{k}{2}t^2e_6 - tke_{10} + e_{14})\Bigg(\binom{k}{2}t^2\Bigg)\\
&\quad + \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_3 + \binom{k}{2}t^2e_7 - tke_{11} + e_{15}\Bigg)(-tk)\\
&\quad + \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_4 + \binom{k}{2}t^2e_8 - tke_{12} + e_{16}.
\end{aligned} \tag{44}
$$

So,

$$
\begin{aligned}
m_1 &= e_1 \geq 0,\\
m_2 &= -tke_1 + e_2 \geq 0,\\
m_3 &= \Bigg(\binom{k}{2}t^2 - k\Bigg)e_1 - tke_2 + e_3 \geq 0,\\
m_4 &= \Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)e_1 + \Bigg(\binom{k}{2}t^2\Bigg)e_2 - -tke_3 + e_4 \geq 0,\\
m_5 &= -tke_1 + e_5 \geq 0,\\
m_6 &= (-tke_1 + e_5)(-tk) + (-tke_2 + e_6) \geq 0,\\
m_7 &= (-tke_1 + e_5)\Bigg(\binom{k}{2}t^2\Bigg) + (-tke_2 + e_6)(-tk)\\
&\quad - tke_3 + e_7 \geq 0,\\
m_8 &= (-tke_1 + e_5)\Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg) + (-tke_2 + e_6)\binom{k}{2}t^2\\
&\quad + (-tke_3 + e_7)(-tk) + (-tke_4 + e_8) \geq 0,\\
m_9 &= \binom{k}{2}t^2e_1 - tke_5 + e_9 \geq 0,\\
m_{10} &= \Bigg(\binom{k}{2}t^2e_1 - tke_5 + e_9\Bigg)(-tk) + \binom{k}{2}t^2e_2\\
&\quad - tke_6 + e_{10} \geq 0,\\
m_{11} &= \Bigg(\binom{k}{2}t^2e_1 - tke_5 + e_9\Bigg)\Bigg(\binom{k}{2}t^2\Bigg)\\
&\quad + \Bigg(\binom{k}{2}t^2e_2 - tke_6 + e_{10}\Bigg)(-tk) + \binom{k}{2}t^2e_3\\
&\quad - tke_7 + e_{11} \geq 0,\\
m_{12} &= \Bigg(\binom{k}{2}t^2e_1 - tke_5 + e_9\Bigg)\Bigg(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\binom{k}{2}t^2e_2 - tke_6 + e_{10}\right)\left(\binom{k}{2}t^2\right)\\
&+\left(\binom{k}{2}t^2e_3 - tke_7 + e_{11}\right)(-tk) + \binom{k}{2}t^2e_4 - tke_8\\
&+ e_{12} \geq 0,\\
m_{13} &= \left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_1 + \binom{k}{2}t^2e_5 - tke_9 + e_{13} \geq 0,\\
m_{14} &= \left(\left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_1 + \binom{k}{2}t^2e_5 - tke_9 + e_{13}\right)\\
&\cdot (-tk) + \left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_2 + \binom{k}{2}t^2e_6\\
&- tke_{10} + e_{14} \geq 0,\\
m_{15} &= \left(\left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_1 + \binom{k}{2}t^2e_5 - tke_9 + e_{13}\right)\\
&\cdot \binom{k}{2}t^2 + \left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_2\\
&+ \binom{k}{2}t^2e_6 - tke_{10} + e_{14}\Bigg)(-tk)\\
&+ \left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_3 + \binom{k}{2}t^2e_7 - tke_{11} + e_{15} \geq 0,\\
m_{16} &= \left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_1 + \binom{k}{2}t^2e_5 - tke_9 + e_{13}\Bigg)\\
&\cdot \left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right) + \left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_2\\
&+ \binom{k}{2}t^2e_6 - tke_{10} + e_{14}\Bigg)\left(\binom{k}{2}t^2\right)\\
&+ \left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_3 + \binom{k}{2}t^2e_7 - tke_{11} + e_{15}\Bigg)\\
&\cdot (-tk) + \left(-\sum_{i=3}^{k}\binom{k}{2}t^3 - k\right)e_4\\
&+ \binom{k}{2}t^2e_8 - tke_{12} + e_{16} \geq 0.
\end{aligned}
\tag{45}
$$

Therefore, we get

$$
\begin{aligned}
&e_1 \geq 0, \quad \frac{e_1}{e_2} \leq \frac{1}{tk},\\
&\frac{e_1}{e_3} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2\right)}, \quad \frac{e_1}{e_4} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 + k - t^3k^3},\\
&\frac{e_1}{e_5} \leq \frac{1}{tk}, \quad \frac{e_2}{e_6} \leq \frac{1}{tk},\\
&\frac{e_3}{e_7} \leq \frac{1}{tk}, \quad \frac{e_4}{e_8} \leq \frac{1}{tk},\\
&\frac{e_1}{e_9} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2\right)}, \quad \frac{e_3}{e_{11}} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2\right)},\\
&\frac{e_4}{e_8} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2\right)}, \quad \frac{e_1}{e_{13}} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 + k - t^3k^3},\\
&\frac{e_2}{e_{14}} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 + k - t^3k^3}, \quad \frac{e_3}{e_{15}} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 + k - t^3k^3},\\
&\frac{e_4}{e_{16}} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 + k - t^3k^3}.
\end{aligned}
\tag{46}
$$

In a similar way, we obtain the following relations for $p \geq 3$:

$$e_1 \geq 0, \quad \frac{e_1}{e_2} \leq \frac{1}{tk}, \tag{47}$$

$$\frac{e_1}{e_3} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2\right)}, \quad \frac{e_1}{e_4} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 - t^3k^3} \tag{48}$$

$$\frac{e_1}{e_5} \leq \frac{1}{tk}, \quad \frac{e_2}{e_6} \leq \frac{1}{tk}, \tag{49}$$

$$\frac{e_3}{e_7} \leq \frac{1}{tk}, \quad \frac{e_4}{e_8} \leq \frac{1}{tk}, \tag{50}$$

$$\frac{e_1}{e_9} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2\right)}, \quad \frac{e_3}{e_{11}} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2\right)}, \tag{51}$$

$$\frac{e_4}{e_8} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2\right)}, \quad \frac{e_1}{e_{13}} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 - t^3k^3}, \tag{52}$$

$$\frac{e_2}{e_{14}} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 - t^3k^3}, \quad \frac{e_3}{e_{15}} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 - t^3k^3}, \tag{53}$$

$$\frac{e_4}{e_{16}} \leq \frac{1}{\sum_{i=3}^{k}\binom{k}{2}t^3 - t^3k^3}. \tag{54}$$

We encounter a question about relations (35)–(54). How large should k be to claim that inequalities (35)–(54) can be converted into equalities? We will give an approximate answer to it by two examples.

Example 3. For $p = 1, t = 5$ and an initial message $M_{2\times 2}$, we have

$$M = \begin{bmatrix} 3 & 5 \\ 8 & 20 \end{bmatrix}. \tag{55}$$

We consider $k = 1$; then,

$$E = \left(L^1(5,2)\right)^1 \times M \times \left(L^1(5,2)\right)^1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 8 & 20 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 20 \\ 24 & 146 \end{bmatrix}. \tag{56}$$

Here, we have the following:

$$\begin{cases} 3, & 320 \simeq 0.015, \\ \dfrac{3}{24} \simeq 0.125, & \dfrac{3}{146} \simeq 0.020. \end{cases} \tag{57}$$

From (35), we have $1/tk = 1/5 = 0.2$ and $1/t^2k^2 = 1/25 = 0.04$.

Now, for $k = 100$, we have

$$E = \left(L^1(5,2)\right)^{100} \times M \times \left(U^1(5,2)\right)^{100} = \begin{bmatrix} 1 & 0 \\ 500 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 8 & 20 \end{bmatrix}\begin{bmatrix} 1 & 500 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1005 \\ 1508 & 756520 \end{bmatrix}. \tag{58}$$

In this case,

$$\begin{cases} 3, & \dfrac{3}{1005} \simeq 0.0029, \\ \dfrac{3}{1508} \simeq 0.0019, & \dfrac{3}{756520} \simeq 0.0000039. \end{cases} \tag{59}$$

Since $1/tk = 1/500 = 0.002$ and $1/t^2k^2 = 1/250000 = 0.000004$, we have the following relations for $k = 100$:

$$\frac{e_1}{e_2} \simeq \frac{1}{tk}, \quad \frac{e_1}{e_3} \simeq \frac{1}{tk}, \quad \frac{e_1}{e_4} \simeq \frac{1}{t^2k^2}. \tag{60}$$

Example 4. For $p = 1, t = 3$ and an initial message as

$$M = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & 3 \\ 4 & 8 & 9 \end{bmatrix}, \tag{61}$$

we consider $k = 2$; then,

$$\begin{aligned} E &= \left(L^1(3,3)\right)^2 \times M \times \left(U^1(3,3)\right)^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 29 & 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & 3 \\ 4 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 6 & 29 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 11 & 66 \\ 8 & 80 & 349 \\ 45 & 435 & 2525 \end{bmatrix}. \end{aligned} \tag{62}$$

In this case, we have the following values:

$$\begin{cases} 1, & \dfrac{1}{11} \simeq 0.09, & \dfrac{1}{66} \simeq 0.015, \\ \dfrac{1}{8} \simeq 0.125, & \dfrac{1}{80} \simeq 0.0125, & \dfrac{1}{349} \simeq 0.0028, \\ \dfrac{1}{45} \simeq 0.022, & \dfrac{1}{435} \simeq 0.00229, & \dfrac{1}{2525} \simeq 0.000396. \end{cases} \tag{63}$$

From (39)–(42), we have

$$\frac{e_1}{e_2} \leq \frac{1}{tk} = \frac{1}{6} \simeq 0.166,$$

$$\frac{e_1}{e_3} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2 - k\right)} = \frac{1}{29} \simeq 0.0344,$$

$$\frac{e_1}{e_4} \leq \frac{1}{tk} = \frac{1}{6} \simeq 0.166,$$

$$\frac{e_1}{e_5} \leq \frac{1}{t^2k^2} = \frac{1}{36} \simeq 0.0277,$$

$$\frac{e_1}{e_6} \leq \frac{1}{t^3k^3 - \left(\binom{k}{2}kt^2 - k^2t\right)} = \frac{1}{186} \simeq 0.0053,$$

$$\frac{e_1}{e_7} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2}t^2 - k\right)} = \frac{1}{29} \simeq 0.0344,$$

$$\frac{e_1}{e_8} \leq \frac{1}{t^3k^3 - \left(\binom{k}{2} kt^2 - k^2 t\right)} = \frac{1}{186} \simeq 0.0053,$$
$$\frac{e_1}{e_9} \leq \frac{1}{\left(t^2k^2 + k - \left(\binom{k}{2} t^2\right)\right)^2} = \frac{1}{29^2} \simeq 0.001189. \tag{64}$$

Now, let $k = 100$. Then, we get

$$E = \left(L^1(3,3)\right)^{100} \times M \times \left(U^1(3,3)\right)^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 5450 & 200 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & 3 \\ 4 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 200 & 5450 \\ 0 & 1 & 200 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 205 & 6457 \\ 202 & 41402 & 1302703 \\ 5854 & 1198458 & 324847519 \end{bmatrix}. \tag{65}$$

In this case, we obtain the following values:

$$\begin{cases} 1, & \frac{1}{205} \simeq 0.0048, & \frac{1}{6457} \simeq 0.00015, \\ \frac{1}{202} \simeq 0.0049, & \frac{1}{41402} \simeq 0.000024, & \frac{1}{1302703} \simeq 0.00000076, \\ \frac{1}{5854} \simeq 0.00017, & \frac{1}{1198458} \simeq 0.00000083, & \frac{1}{324847519} \simeq 0.000003. \end{cases} \tag{66}$$

From (39)–(42), we have

$$\frac{e_1}{e_2} \leq \frac{1}{tk} = \frac{1}{200} \simeq 0.005,$$
$$\frac{e_1}{e_3} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2} t^2 - k\right)} = \frac{1}{20300} \simeq 0.000049,$$
$$\frac{e_1}{e_4} \leq \frac{1}{tk} = \frac{1}{200} \simeq 0.005,$$
$$\frac{e_1}{e_5} \leq \frac{1}{t^2k^2} = \frac{1}{40000} \simeq 0.000025,$$
$$\frac{e_1}{e_6} \leq \frac{1}{t^3k^3 - \left(\binom{k}{2} kt^2 - k^2 t\right)} = \frac{1}{204000} \simeq 0.0000049,$$
$$\frac{e_1}{e_7} \leq \frac{1}{t^2k^2 - \left(\binom{k}{2} t^2 - k\right)} = \frac{1}{20300} \simeq 0.000049,$$
$$\frac{e_1}{e_8} \leq \frac{1}{t^3k^3 - \left(\binom{k}{2} tk^2 - k^2 t\right)} = \frac{1}{2040000} \simeq 0.0000049,$$
$$\frac{e_1}{e_9} \leq \frac{1}{t^2k^2 + k - \left(\binom{k}{2} t^2\right)^2} = \frac{1}{20300^2} \simeq 0.0000024. \tag{67}$$

Therefore, by considering t and the matrix M, we can find $k = 100$ is an appropriate value.

Now, we calculate the error detection for the t-extension of the p-Fibonacci Pascal decoding. Error detection and correction of the code message E are the most important aim of the coding theory. First, we should determine the error detection in the code message E. For this, using the property of the determinant of a matrix, we can check the transmitted message E in the communication channel. We have

$$E = \left(\left(L^p(t,n)\right)^k \times M \times \left(U^p(t,n)\right)^k\right),$$
$$\det E = \det\left(\left(L^p(t,n)\right)^k \times M \times \left(U^p(t,n)\right)^k\right) = \det M. \tag{68}$$

Therefore, it is clear that the determinant of the initial message M is connected with the determinant of the code message E. So, we obtain the determinant of the matrix M. det (M) treats as a controller of entries of the code matrix E received from the communication channel. After receiving the code matrix E and computing the determinant of M, we will compute the determinant of E. Then, we will compare them together. If $\det(E) = \det(M)$, this means that the matrix E has passed from the communication channel without error. Otherwise, according to the matrix E of the order $n \times n (1 \leq n \leq 4)$, we have "single," "double," $\cdots$, "n^2-fold" errors. Thus, we get

$$1C_{n^2} + 2C_{n^2} + \cdots + n^2 C_{n^2} = 2^{n^2} - 1. \tag{69}$$

Now, we will compute the error correction. Suppose $p = 1$ and $n = 2$. Hence, there exists only one error in the matrix E received from the communication channel. The four variants of the single errors in the code matrix E are as follows:

$$1)\begin{bmatrix} a & e_2 \\ e_3 & e_4 \end{bmatrix} \quad 2)\begin{bmatrix} e_1 & b \\ e_3 & e_4 \end{bmatrix}, \quad 3)\begin{bmatrix} e_1 & e_2 \\ c & e_4 \end{bmatrix}, \quad 4)\begin{bmatrix} e_1 & e_2 \\ e_3 & d \end{bmatrix}, \tag{70}$$

where a, b, c, and d are the destroyed elements. By algebraic equations and Lemma 10, we obtain

$$a = \frac{\det M + e_2 e_3}{e_4}, \tag{71}$$

$$b = \frac{-\det M + e_1 e_4}{e_3}, \tag{72}$$

$$c = \frac{-\det M + e_1 e_4}{e_2} \tag{73}$$

$$d = \frac{\det M + e_2 e_3}{e_1}. \tag{74}$$

From relations (71)–(74) and (35), we calculate the destroyed element in the code matrix E.

In a similar way, we can correct "double" and "triple" errors. For example, we consider the following case of double errors in the matrix E,

$$\begin{bmatrix} g & h \\ e_3 & e_4 \end{bmatrix}, \tag{75}$$

where g and h are the destroyed elements. We have

$$g e_4 - h e_3 = \det M. \tag{76}$$

Similarly, we compute the destroyed element in the code matrix E. So, there are $2^{2^2} - 1 = 15$ errors for $n = 2, p = 1$, and we can correct all cases of this method except the fourfold. Thus, the correctable possibility of it is equal to $14/15 = 0.9333 = \%93.33$.

Similarly, there are $2^{3^2} - 1 = 511$ errors for $n = 3, p = 1$. Since $\det E = \det M$ and from (40), we can correct up to eight cases of this method except the ninefold. Therefore, we get that correctable possibility of the method is equal to $510/511 = 0.9980 = \%99.80$. And finally, we obtain $(2^{16} - 2)/(2^{16} - 1) \simeq 1 = \%100$ for $n = 4, p \geq 3$.

The t-extension of the p-Fibonacci Pascal coding method has a high correction ability in comparison to the classical (algebraic) coding method. The reason is the use of matrix theory to get error-correction codes in this coding method, while in algebraic coding method there are very small information elements and bits and their combinations are the objects of detection and correction. For example, we compare the t-extension of the p-Fibonacci Pascal coding method to the Hamming coding by an example and show the correctable ability of the error is much greater in the t-extension of the p-Fibonacci Pascal coding method. For $n = 15$ and $k = 10$, we consider Hamming code method. We will obtain the correctable ability of the error. There is $2^{10}(2^{15} - 2^{10}) = 32505856$ error messages and the number $2^{15} - 2^{10} = 31744$ can just be corrected; then, the correctable ability of the error equals

$$\frac{31744}{32505856} = 0.0009765 = \%0.09765. \tag{77}$$

Besides, the correctable ability of the error in the t-extension of the 3-Fibonacci Pascal coding method is % 99.80. Thus, the correctable ability of the k-Fibonacci Pascal is 10000 times more than Hamming one.

4. Conclusion

In this paper, we obtained the k-th power of the t-extension of the p-Fibonacci matrix $(L^p(t, n)^k)$ and the inverse of them $(L^p(t, n)^{-k})$ for $(2 \leq n \leq 4)$. Then, using them, we introduced a new coding/decoding method. The t-extension of the p-Fibonacci Pascal coding method is the main application of the matrices $L^p(t, n)^k$ and $L^p(t, n)^{-k}$. The t-extension of the p-Fibonacci Pascal matrix coding/decoding was calculated very quickly by computer. Also, the correcting and detection abilities of this coding method were very high in comparison with a classical algebraic coding/decoding method.

Data Availability

There are no applications, analysis, or generation during the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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