British Journal of Mathematics & Computer Science
4(8): 1091-1123, 2014

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Weighted Hesitant Fuzzy Sets and Their Application to Multi-Criteria Decision Making

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Authors' Contributions

ZZ designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. CW managed the analyses of the study and the literature searches. All authors read and approved the final manuscript.

Original Research Article

Received: 18 December 2013
Accepted: 01 February 2014
Published: 17 February 2014

Abstract

Aims: The aim of this paper is to investigate weighted hesitant fuzzy sets and their application to multi-criteria decision making.
Study Design: This paper puts forward the concept of a weighted hesitant fuzzy set (WHFS), in which several possible membership degrees of each element have different weights. Archimedean t-conorm and t-norm provide a generalization of a variety of other t-conorms and t-norms that include as special cases Algebraic, Einstein, Hamacher and Frank t-conorms and t-norms.
Place and Duration of Study: Hesitant fuzzy set, permitting the membership degree of an element to be a set of several possible values, can be referred to as an efficient mathematical tool for modeling people's hesitancy in daily life. It is noted that several possible membership degrees of each element in the hesitant fuzzy set are of equal importance, but in many practical problems, especially in multi-criteria decision making, the weights of several possible membership degrees of each element should be taken into account.
Methodology: In this paper, based on Archimedean t-conorm and t-norm, we present some operations on weighted hesitant fuzzy sets (WHESs), and based on which, we develop two weighted hesitant fuzzy aggregation operators for aggregating weighted hesitant fuzzy information. Furthermore, some desired properties and special cases of the developed operators are discussed in detail.
Results: We develop an approach for multi-criteria decision making under weighted hesitant fuzzy environment.

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Conclusion: An illustrative example is provided to show the effectiveness and practicality of the proposed operators and approach.

Keywords: Multi-criteria decision making; hesitant fuzzy sets; weighted hesitant fuzzy sets; archimedean t-conorm and t-norm; weighted hesitant fuzzy aggregation operator.

1 Introduction

Due to the fact that when defining the membership degree of an element to a set, the difficulty of establishing the membership degree is not because we have a margin of error (as in intuitionstic fuzzy set [1], interval-valued fuzzy set [2], or interval-valued intuitionistic fuzzy set [3]) or some possibility distribution (as in type-2 fuzzy set [4]) on the possible values, but because we have some possible values, Torra [5] defined the hesitant fuzzy sets (HFS) to permit the membership degree of an element to a set represented as several possible values between 0 and 1. The HFS can be used to efficiently manage the situation where people hesitate between several possible values to express their opinions. Since it was introduced, HFS has attracted much attention. Torra and Narukawa [6] first applied hesitant fuzzy sets (HFSs) to decision making. Xu and Xia [7,8] proposed a lot of distance measures, similarity measures, and correlation measures for HFSs. Farhadinia [9] investigated the relationship between the entropy, the similarity measure, and the distance measure for HFSs and interval-valued hesitant fuzzy sets (IVHFSs) [10,11]. Peng et al. [12] presented a generalized hesitant fuzzy synergetic weighted distance (GHFSWD) measure based on the generalized hesitant fuzzy weighted distance (GHFWD) measure and the generalized hesitant fuzzy ordered weighted distance (GHFOWD) measure proposed in [7]. Qian et al. [13] extended hesitant fuzzy sets by intuitionistic fuzzy sets and referred to them as generalized hesitant fuzzy sets.

It should be noted that only several possible values are involved in the classical HFS, but the importance of each possible value is not emphasized. Nevertheless, in many practical situations, especially in multi-criteria decision making, several possible values usually have different importance and thus need to be assigned different weights. For example, to get a reasonable decision result, ten decision makers who are very familiar with this area are invited to estimate the degree that an alternative satisfies an attribute. Suppose there are four cases, four decision makers provide 0.8, three decision makers provide 0.7, two decision makers provide 0.6, and one decision maker provides 0.5, and these ten decision makers cannot persuade each other to change their opinions. In [7,14,15,16,17], the authors do not consider the importance of all of the possible values for an alternative under an attribute and allow these values repeated many times appear only once. According to [7,14,15,16,17], the degree that the alternative satisfies the attribute is represented by a HFS (0.5, 0.6, 0.7, 0.8), which is somewhat inconsistent with our intuition because these values repeated many times at least denote strength of the decision makers' preferences. According to the strategy given in [7,14,15,16,17], more experts may not contribute to more reasonable decision results. When we consider a multiple attribute group decision making (MAGDM) problem, if two or more decision makers who are familiar with this area give the same preferences, then their preferences will be close to group preference. In such cases, the value repeated many times may be more important than the one repeated only one time. Therefore, the importance of all of the possible membership degrees should be attached to the construction of the HFS. To do it, in this paper, we introduce the concept of a weighted hesitant fuzzy set (WHFS), which is a new generalization of the classical hesitant fuzzy set by adding the weight information to the classical hesitant fuzzy set. In the WHFS, the importance of all of the possible membership degrees is taken into account and the weight information is associated with all of the possible membership degrees. Thus, the WHFS can contain more information than the classical hesitant fuzzy set and can help the decision makers get more accurate, reasonable, and reliable decision results than the classical hesitant fuzzy set. In the previous example, the degree that the alternative satisfies the attribute can be represented by a WHFS $\{(0.5, 0.1), (0.6, 0.2), (0.7, 0.3), (0.8, 0.4)\}$.

In order to aggregate hesitant fuzzy information, Xia and Xu [15] proposed some Algebraic t-conorm and t-norm based operational laws for HFSs, based on which, a variety of hesitant fuzzy aggregation operators have been developed in recent years. For example, Xia and Xu [15] developed the hesitant fuzzy weighted averaging (HFWA) operator, the hesitant fuzzy weighted geometric (HFWG) operator, the generalized hesitant fuzzy weighted averaging (GHFWA) operator, the generalized hesitant fuzzy weighted geometric (GHFWG) operator, the hesitant fuzzy ordered weighted averaging (HFOWA) operator, the hesitant fuzzy ordered weighted geometric (HFOWG) operator, the generalized hesitant fuzzy ordered weighted averaging (GHFOWA) operator, the generalized hesitant fuzzy ordered weighted geometric (GHFOWG) operator, the hesitant fuzzy hybrid averaging (HFHA) operator, the hesitant fuzzy hybrid geometric (HFHG) operator, the generalized hesitant fuzzy hybrid averaging (HFHA) operator, and the generalized hesitant fuzzy hybrid geometric (GHFHG) operator. Xia et al. [18] proposed some new hesitant fuzzy aggregation operators, such as the quasi hesitant fuzzy weighted aggregation (QHFWA) operator, the hesitant fuzzy modular weighted averaging(QHFWA) operator, the hesitant fuzzy modular weighted geometric (HFMWG) operator, the quasi hesitant fuzzy ordered weighted aggregation (QHFOWA) operator, the hesitant fuzzy modular ordered weighted averaging (QHFOWA) operator, the hesitant fuzzy modular ordered weighted geometric (HFMWG) operator, the induced quasi hesitant fuzzy ordered weighted aggregation (IQHFOWA) operator, the induced hesitant fuzzy modular ordered weighted averaging (IHFMOWA) operator, and the induced hesitant fuzzy modular ordered weighted geometric (IHFMWG) operator. By extending the Bonferroni mean (BM) [19] to hesitant fuzzy environments, Zhu and Xu [20] developed the hesitant fuzzy Bonferroni means (HFBMs) and the weighted hesitant fuzzy Bonferroni mean (WHFBM). By extending the geometric Bonferroni mean (BM) [21] to hesitant fuzzy environments, Zhu et al. [22] proposed the hesitant fuzzy geometric Bonferroni mean (HFGBM) and the weighted hesitant fuzzy Choquet geometric Bonferroni mean (WHFCGBM). In order to consider the relationship between the hesitant fuzzy input arguments, Zhang [23] developed several new hesitant fuzzy aggregation operators, including the hesitant fuzzy power average (HFPA) operator, the hesitant fuzzy power geometric (HFPG) operator, the generalized hesitant fuzzy power average (GHFPA) operator, the generalized hesitant fuzzy power geometric (GHFPG)operator, the weighted the generalized hesitant fuzzy power average (WGHFPA) operator, the weighted generalized hesitant fuzzy power geometric (WGHFPG) operator, the hesitant fuzzy power ordered weighted average (HFPOWA) operator, the hesitant fuzzy power ordered weighted geometric(HFPOWG) operator, the generalized hesitant fuzzy power ordered weighted average (GHFPOWA) operator, and the generalized hesitant fuzzy power ordered weighted geometric (GHFPOWG) operator. Wei and Zhao [24] introduced some operations on hesitant interval-valued fuzzy sets (HIVFSs) based on Einstein t-conorm and t-norm, and based on which, developed some induced hesitant interval-valued fuzzy Einstein aggregation operators for aggregating hesitant interval-valued fuzzy information.

Archimedean t-conorm and t-norm [25,26] are generalizations of lots of other t-conorms and t-norms, such as Algebraic, Einstein, Hamacher and Frank t-conorms and t-norms [27]. Based on Archimedean t-conorm and t-norm, Beliakov et al. [28] gave some operations about intuitionistic fuzzy sets (IFSs). Xia et al. [27] further gave some other operations on IFSs, and proposed some specific intuitionistic fuzzy aggregation operators. Motivated by the work of Beliakov et al. [28] and Xia et al. [27], this paper proposes some Archimedean t-conorm and t-norm based operation laws on weighted hesitant fuzzy sets (WHFSs), investigates their properties, and based on which, develops two Archimedean t-conorm and t-norm based weighted hesitant fuzzy aggregation operators, including the Archimedean t-conorm and t-norm based weighted hesitant fuzzy weighted averaging (ATS-WHFWA) operator and the Archimedean t-conorm and t-norm based weighted hesitant fuzzy weighted geometric (ATS-WHFWG) operator. Moreover, we study some desired properties of the new operators and give their special cases, such as the weighted hesitant fuzzy weighted averaging (WHFWA) operator, the weighted hesitant fuzzy Einstein weighted averaging (WHFEWA) operator, the weighted hesitant fuzzy Hammer weighted averaging (WHFHWA) operator, the weighted hesitant fuzzy Frank weighted averaging (WHFFWA) operator, the weighted hesitant fuzzy weighted geometric (WHFWG) operator, the weighted hesitant fuzzy Einstein weighted geometric (WHFEWG) operator, the weighted hesitant fuzzy Hammer weighted geometric (WHFHWG) operator, and the weighted hesitant fuzzy Frank weighted geometric (WHFFWG) operator. Finally, we develop an approach for multi-criteria decision making under weighted hesitant fuzzy environment, and provide a numerical example to illustrate the proposed approach.

This paper is organized as follows. Section 2 introduces some basic concepts of hesitant fuzzy sets and Archimedean t-conorm and t-norm. In Section 3, we define the concept of weighted hesitant fuzzy sets (WHFSs) and introduce some operational laws for them based on Archimedean t-conorm and t-norm. Section 4 proposes two Archimedean t-conorm and t-norm based weighted hesitant fuzzy aggregation operators for aggregation weighted hesitant fuzzy information. Some desired properties and special cases of the proposed operators are also investigated in this section. In the sequel, Section 5 develops an approach to multi-criteria decision making under weighted hesitant fuzzy environment and gives a practical example to illustrate the developed approach. The final section offers some concluding remarks.

2 Preliminaries

In this section, we will give a brief introduction of hesitant fuzzy sets [5] and Archimedean t-conorm and t-norm [25,26].

Definition 2.1 [5]. Let X be a fixed set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of $[0,1]$.

Xia and Xu [15] expressed a HFS by the following form:

$$E=\left\{\langle x,h_E(x)\rangle \middle| x\in X\right\} \tag{1}$$

where $h_E(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x\in X$ to the set E. For convenience, Xia and Xu [15] called $h=h_E(x)$ a hesitant fuzzy element (HFE) and H the set of all hesitant fuzzy elements (HFEs).

Assume three HFEs represented by h, h_1 and h_2, Torra [5] defined some operations on them, which can be described as:

$$h^c = \{1-\gamma | \gamma \in h\};\tag{2}$$

$$h_1 \cup h_2 = \{\gamma_1 \vee \gamma_2 | \gamma_1 \in h_1, \gamma_2 \in h_2\};\tag{3}$$

$$h_1 \cap h_2 = \{\gamma_1 \wedge \gamma_2 | \gamma_1 \in h_1, \gamma_2 \in h_2\}.\tag{4}$$

Definition 2.2 [25,26]. A function $T:[0,1]\times[0,1]\to[0,1]$ is called a t-norm if it satisfies the following four conditions:

(1) $T(1,a)=a$, for all $a\in[0,1]$.
(2) $T(a,b)=T(b,a)$, for all $a,b\in[0,1]$.
(3) $T(a,T(b,c))=T(T(a,b),c)$, for all $a,b,c\in[0,1]$.
(4) If $a\le a'$ and $b\le b'$ for all $a,a',b,b'\in[0,1]$, then $T(a,b)\le T(a',b')$.

Definition 2.3 [25,26]. A function $S:[0,1]\times[0,1]\to[0,1]$ is called a t-conorm if it satisfies the following four conditions:

(1) $S(0,a)=a$, for all $a\in[0,1]$.
(2) $S(a,b)=S(b,a)$, for all $a,b\in[0,1]$.
(3) $S(a,S(b,c))=S(S(a,b),c)$, for all $a,b,c\in[0,1]$.
(4) If $a\le a'$ and $b\le b'$ for all $a,a',b,b'\in[0,1]$, then $S(a,b)\le S(a',b')$.

Definition 2.4 [25,26]. A t-norm function $T(a,b)$ is called Archimedean t-norm if it is continuous and $T(a,a)<a$ for all $a\in(0,1)$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $a,b\in(0,1)$.

Definition 2.5 [25,26]. A t-conorm function $S(a,b)$ is called Archimedean t-conorm if it is continuous and $S(a,a)>a$ for all $a\in(0,1)$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $a,b\in(0,1)$.

It is well known [29] that a strict Archimedean t-norm $T(a,b)$ is expressed via its additive generator g as $T(a,b)=g^{-1}(g(a)+g(b))$, where $g:[0,1]\to[0,+\infty]$ is a strictly decreasing function such that $g(1)=0$. A dual Archimedean t-conorm $S(a,b)$ is expressed as $S(a,b)=f^{-1}(f(a)+f(b))$ with $f(t)=g(1-t)$.

3 Weighted Hesitant Fuzzy Sets (WHFSS) and Weighted Hesitant Fuzzy Elements (WHFES)

Considering that the classical hesitant fuzzy set does not involve the importance of all of the possible membership degrees of each element, in this section, we will propose a new concept of weighted hesitant fuzzy set by assigning a weight vector to all of the possible membership degrees of each element.

Definition 3.1. Let X be a reference set, a weighted hesitant fuzzy set (WHFS) on X is defined as:

$$\tilde{A}=\left\{\left\langle x,\tilde{h}_{\tilde{A}}(x)\right\rangle \middle| x\in X\right\}=\left\{\left\langle x,\bigcup_{\gamma\in h_{\tilde{A}}(x)}\left\{\left(\gamma,w_{x\gamma}\right)\right\}\right\rangle \middle| x\in X\right\} \tag{5}$$

where $h_{\tilde{A}}(x)$ is a set of some different values in $[0,1]$, denoting all possible membership degrees of the element $x\in X$ to the set $\tilde{A}$, $w_{x\gamma}$ is the weight of γ, $w_{x\gamma}\in[0,1]$, and $\sum_{\gamma\in h_{\tilde{A}}(x)} w_{x\gamma}=1$ for any $x\in X$.

For convenience, we call $\tilde{h}=\bigcup_{\gamma\in h}\left\{\left(\gamma,w_{\gamma}\right)\right\}$ a weighted hesitant fuzzy element (WHFE), where h is a set of some different membership degrees in $[0,1]$, w_{γ} is the weight of γ, $w_{\gamma}\in[0,1]$ for any $\gamma\in h$, and $\sum_{\gamma\in h} w_{\gamma}=1$. Let $\tilde{H}$ denote the set of all weighted hesitant fuzzy elements (WHFEs).

Let $\tilde{A}=\left\{\left\langle x,\tilde{h}_{\tilde{A}}(x)\right\rangle \middle| x\in X\right\}=\left\{\left\langle x,\bigcup_{\gamma\in h_{\tilde{A}}(x)}\left\{\left(\gamma,w_{x\gamma}\right)\right\}\right\rangle \middle| x\in X\right\}$ be a WHFS. If for any $x\in X$ and $\gamma\in h_{\tilde{A}}(x)$, $w_{x\gamma}=\frac{1}{\#h_{\tilde{A}}(x)}$ ($\#h_{\tilde{A}}(x)$ is the number of the elements in $h_{\tilde{A}}(x)$), then $\tilde{A}$ reduces to a HFS. Let $\tilde{h}=\bigcup_{\gamma\in h}\left\{\left(\gamma,w_{\gamma}\right)\right\}$ be a WHFE. If $w_{\gamma}=\frac{1}{\#h}$, then $\tilde{h}$ reduces to a HFE.

By Definition 3.1, the WHFS extends the HFS to contain several membership degrees and their corresponding weights. The difference between the HFS and WHFS is that the former assumes that the possible membership degrees of each element are of equal importance, while the latter assigns different weights to different membership degrees. Thus, compared with the HFS, the WHFS can depict human uncertainty more objectively and precisely.

In the following, we illustrate how to construct a WHFE. Suppose that l experts are required to evaluate the membership degree of the element x in the set $\tilde{A}$. l_1 experts provide γ_1, l_2 experts provide γ_2, $\cdots$, and l_k experts provide γ_k, where $\sum_{i=1}^{k} l_k = l$. Assume that these l experts cannot persuade each other to change their opinions. In such cases, the membership degree of the element x in the set $\tilde{A}$ has k possible values γ_1, γ_2, $\cdots$, and γ_k. The weights of γ_i ($i = 1, 2, \cdots, k$) are $w_{\gamma_i} = \frac{l_i}{l}$ ($i = 1, 2, \cdots, k$). Thus, the membership degree of the element x in the set $\tilde{A}$ can be represented by a WHFE $\tilde{h} = \left\{ \left(\gamma_1, \frac{l_1}{l} \right), \left(\gamma_2, \frac{l_2}{l} \right), \cdots, \left(\gamma_k, \frac{l_k}{l} \right) \right\}$. Based on the above analysis, we can see that constructing of a WHFE consists of two steps: (1) collecting different possible membership degree values into a HFE; (2) assigning the weights to these different membership degree values.

The WHFS is an efficient tool to represent situations in which several different membership functions for a fuzzy set are possible and different membership functions have different weights. It is particularly suitable to address the hesitancy and uncertainty that are quite usual in real world decision making problems.

Example 3.1. Let $X = \{x_1, x_2, x_3\}$,
$\tilde{A} = \left\{ \left\langle x_1, \{(0.5, 0.3), (0.6, 0.7)\} \right\rangle, \left\langle x_2, \{(0.1, 0.5), (0.3, 0.2), (0.4, 0.3)\} \right\rangle, \left\langle x_3, \{(0.7, 0.5), (0.9, 0.5)\} \right\rangle \right\}$, and $\tilde{h} = \{(0.1, 0.5), (0.3, 0.2), (0.4, 0.3)\}$. Then, $\tilde{A}$ is a WHFS on X and $\tilde{h}$ is a WHFE.

To compare two WHFEs, we define the following comparison laws:

Definition 3.2. For a WHFE $\tilde{h} = \bigcup_{\gamma \in h} \{(\gamma, w_\gamma)\}$, $s(\tilde{h}) = \sum_{\gamma \in h} (w_\gamma \cdot \gamma)$ is called the score function of $\tilde{h}$. For two WHFEs $\tilde{h}_1$ and $\tilde{h}_2$, if $s(\tilde{h}_1) > s(\tilde{h}_2)$, then $\tilde{h}_1 > \tilde{h}_2$; if $s(\tilde{h}_1) = s(\tilde{h}_2)$, then $\tilde{h}_1 = \tilde{h}_2$.

Given three WHFEs represented by $\tilde{h} = \bigcup_{\gamma \in h} \{(\gamma, w_\gamma)\}$, $\tilde{h}_1 = \bigcup_{\gamma_1 \in h_1} \{(\gamma_1, w_{1\gamma_1})\}$, and $\tilde{h}_2 = \bigcup_{\gamma_2 \in h_2} \{(\gamma_2, w_{2\gamma_2})\}$, we define some basic operations on them as below:

$$\tilde{h}^c = \bigcup_{\gamma \in h} \{(1 - \gamma, w_\gamma)\}; \tag{6}$$

$$\tilde{h}_1 \cup \tilde{h}_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{(\gamma_1 \vee \gamma_2, w_{1\gamma_1} \cdot w_{2\gamma_2})\}; \tag{7}$$

$$\tilde{h}_1 \cap \tilde{h}_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \left(\gamma_1 \wedge \gamma_2, w_{1\gamma_1} \cdot w_{2\gamma_2} \right) \right\}. \tag{8}$$

Theorem 3.1. Let $\tilde{h}$, $\tilde{h}_1$, and $\tilde{h}_2$ be three WHFEs. Then, $\tilde{h}^c$, $\tilde{h}_1 \cup \tilde{h}_2$, and $\tilde{h}_1 \cap \tilde{h}_2$ are also WHFEs.

Proof. It is clear that $\tilde{h}^c$ is a WHFE.

According to Definition 3.1, we have

$$\sum_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left(w_{1\gamma_1} \cdot w_{2\gamma_2} \right) = \sum_{\gamma_1 \in h_1} \left(w_{1\gamma_1} \cdot \left(\sum_{\gamma_2 \in h_2} w_{2\gamma_2} \right) \right) = \sum_{\gamma_1 \in h_1} \left(w_{1\gamma_1} \cdot 1 \right) = 1$$

which shows that $\tilde{h}_1 \cup \tilde{h}_2$ is a WHFE.

Similarly, we can conclude that $\tilde{h}_1 \cap \tilde{h}_2$ is also a WHFE.

This completes the proof of Theorem 3.1. □

In the above operations, we allow the membership degrees repeated many times appear only once, whose weight is a sum of the weights of the membership degrees repeated many times.

Example 3.2. Given two WHFEs $\tilde{h}_1$ and $\tilde{h}_2$ as follows:

$$\tilde{h}_1 = \{(0.1, 0.5), (0.3, 0.2), (0.4, 0.3)\}, \quad \tilde{h}_2 = \{(0.7, 0.6), (0.8, 0.4)\}.$$

Then, we have

$$\begin{aligned}
\tilde{h}_1^c &= \{(0.9, 0.5), (0.7, 0.2), (0.6, 0.3)\} \\
\tilde{h}_1 \cup \tilde{h}_2 &= \left\{ \begin{matrix} (0.1 \vee 0.7, 0.5 \cdot 0.6), (0.1 \vee 0.8, 0.5 \cdot 0.4), (0.3 \vee 0.7, 0.2 \cdot 0.6), \\ (0.3 \vee 0.8, 0.2 \cdot 0.4), (0.4 \vee 0.7, 0.3 \cdot 0.6), (0.4 \vee 0.8, 0.3 \cdot 0.4) \end{matrix} \right\} \\
&= \{(0.7, 0.3), (0.8, 0.2), (0.7, 0.12), (0.8, 0.08), (0.7, 0.18), (0.8, 0.12)\} \\
&= \{(0.7, 0.6), (0.8, 0.4)\} \\
\tilde{h} \cap \tilde{h}_2 &= \left\{ \begin{matrix} (0.1 \wedge 0.7, 0.5 \cdot 0.6), (0.1 \wedge 0.8, 0.5 \cdot 0.4), (0.3 \wedge 0.7, 0.2 \cdot 0.6), \\ (0.3 \wedge 0.8, 0.2 \cdot 0.4), (0.4 \wedge 0.7, 0.3 \cdot 0.6), (0.4 \wedge 0.8, 0.3 \cdot 0.4) \end{matrix} \right\} \\
&= \{(0.1, 0.3), (0.1, 0.2), (0.3, 0.12), (0.3, 0.08), (0.4, 0.18), (0.4, 0.12)\} \\
&= \{(0.1, 0.5), (0.3, 0.2), (0.4, 0.3)\}
\end{aligned}$$

In order to aggregate weighted hesitant fuzzy information, we defined some new operations on the WHFEs $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$:

$$\tilde{h}_1 \oplus \tilde{h}_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \left(S(\gamma_1, \gamma_2), w_{1\gamma_1} \cdot w_{2\gamma_2} \right) \right\} = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \left(f^{-1}\left(f(\gamma_1) + f(\gamma_2) \right), w_{1\gamma_1} \cdot w_{2\gamma_2} \right) \right\}; \tag{9}$$

$$\tilde{h}_1 \otimes \tilde{h}_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \left(T(\gamma_1, \gamma_2), w_{1\gamma_1} \cdot w_{2\gamma_2} \right) \right\} = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \left(g^{-1}\left(g(\gamma_1) + g(\gamma_2) \right), w_{1\gamma_1} \cdot w_{2\gamma_2} \right) \right\}; \tag{10}$$

$$\lambda \tilde{h} = \bigcup_{\gamma \in h} \left\{ \left(f^{-1}\left(\lambda f(\gamma) \right), w_\gamma \right) \right\},\ \lambda > 0; \tag{11}$$

$$\tilde{h}^\lambda = \bigcup_{\gamma \in h} \left\{ \left(g^{-1}\left(\lambda g(\gamma) \right), w_\gamma \right) \right\},\ \lambda > 0. \tag{12}$$

Theorem 3.2. For three WHFEs $\tilde{h}$, $\tilde{h}_1$, and $\tilde{h}_2$, we have the following properties:

(1) $\tilde{h}_1 \oplus \tilde{h}_2 = \tilde{h}_2 \oplus \tilde{h}_1$;

(2) $\tilde{h}_1 \otimes \tilde{h}_2 = \tilde{h}_2 \otimes \tilde{h}_1$;

(3) $\lambda\left(\tilde{h}_1 \oplus \tilde{h}_2\right) = \lambda \tilde{h}_1 \oplus \lambda \tilde{h}_2$, $\lambda > 0$;

(4) $\left(\tilde{h}_1 \otimes \tilde{h}_2\right)^\lambda = \tilde{h}_1^\lambda \otimes \tilde{h}_2^\lambda$, $\lambda > 0$;

(5) $\lambda_1 \tilde{h} \oplus \lambda_2 \tilde{h} = \left(\lambda_1 + \lambda_2\right) \tilde{h}$, $\lambda_1, \lambda_2 > 0$;

(6) $\tilde{h}^{\lambda_1} \otimes \tilde{h}^{\lambda_2} = \tilde{h}^{\lambda_1 + \lambda_2}$, $\lambda_1, \lambda_2 > 0$;

(7) $\left(\tilde{h} \oplus \tilde{h}_1\right) \oplus \tilde{h}_2 = \tilde{h} \oplus \left(\tilde{h}_1 \oplus \tilde{h}_2\right)$;

(8) $\left(\tilde{h} \otimes \tilde{h}_1\right) \otimes \tilde{h}_2 = \tilde{h} \otimes \left(\tilde{h}_1 \otimes \tilde{h}_2\right)$;

(9) $\left(\tilde{h}_1 \cup \tilde{h}_2\right)^c = \tilde{h}_1^c \cap \tilde{h}_2^c$;

(10) $\left(\tilde{h}_1 \cap \tilde{h}_2\right)^c = \tilde{h}_1^c \cup \tilde{h}_2^c$;

(11) $\left(\tilde{h}^c\right)^\lambda = \left(\lambda \tilde{h}\right)^c$;

(12) $\lambda\left(\tilde{h}^c\right) = \left(\tilde{h}^\lambda\right)^c$;

(13) $\left(\tilde{h}_1 \oplus \tilde{h}_2\right)^c = \tilde{h}_1^c \otimes \tilde{h}_2^c$;

(14) $\left(\tilde{h}_1 \otimes \tilde{h}_2\right)^c = \tilde{h}_1^c \oplus \tilde{h}_2^c$.

4 Aggregation Operators for Weighted Hesitant Fuzzy Information

In the current section, we will propose several operators for aggregating the weighted hesitant fuzzy information and investigate some properties of these operators.

Definition 4.1. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of WHFEs, and let $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $\tilde{h}_i$ ($i=1,2,\cdots,n$) with $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$. Then, an Archimedean t-conorm and t-norm based weighted hesitant fuzzy weighted averaging (ATS-WHFWA) operator is a mapping $\tilde{H}^n\to\tilde{H}$, where

$$\text{ATS-WHFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\overset{n}{\underset{i=1}{\oplus}}\left(\omega_i\tilde{h}_i\right) \tag{13}$$

Theorem 4.1. Let $\tilde{h}_i=\bigcup_{\gamma_i\in h_i}\left\{\left(\gamma_i,w_{i\gamma_i}\right)\right\}$ ($i=1,2,\cdots,n$) be a collection of WHFEs, and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $\tilde{h}_i$ ($i=1,2,\cdots,n$), where ω_i indicates the importance degree of $\tilde{h}_i$, satisfying $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, then the aggregated value by using the ATS-WHFWA operator is also a WHFE, and

$$\text{ATS-WHFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f(\gamma_i)\right),\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \tag{14}$$

Proof. By using mathematical induction on n: For $n=2$, since

$$\omega_1\tilde{h}_1=\bigcup_{\gamma_1\in h_1}\left\{\left(f^{-1}\left(\omega_1 f(\gamma_1)\right),w_{1\gamma_1}\right)\right\}$$
$$\omega_2\tilde{h}_2=\bigcup_{\gamma_2\in h_2}\left\{\left(f^{-1}\left(\omega_2 f(\gamma_2)\right),w_{2\gamma_2}\right)\right\}$$

we have

$$\begin{aligned}\omega_1\tilde{h}_1\oplus\omega_2\tilde{h}_2&=\left(\bigcup_{\gamma_1\in h_1}\left\{\left(f^{-1}\left(\omega_1 f(\gamma_1)\right),w_{1\gamma_1}\right)\right\}\right)\oplus\left(\bigcup_{\gamma_2\in h_2}\left\{\left(f^{-1}\left(\omega_2 f(\gamma_2)\right),w_{2\gamma_2}\right)\right\}\right)\\&=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\left(f^{-1}\left(f\left(f^{-1}\left(\omega_1 f(\gamma_1)\right)\right)+f\left(f^{-1}\left(\omega_2 f(\gamma_2)\right)\right)\right),w_{1\gamma_1}\cdot w_{2\gamma_2}\right)\right\}\\&=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\left(f^{-1}\left(\omega_1 f(\gamma_1)+\omega_2 f(\gamma_2)\right),w_{1\gamma_1}\cdot w_{2\gamma_2}\right)\right\}\end{aligned}$$

That is, the Eq. (14) holds for $n=2$. Suppose that the Eq. (14) holds for $n=k$, i.e.,

$$\overset{k}{\underset{i=1}{\oplus}}\left(\omega_i\tilde{h}_i\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k}\left\{\left(f^{-1}\left(\sum_{i=1}^{k}\omega_i f\left(\gamma_i\right)\right),\prod_{i=1}^{k}w_{i\gamma_i}\right)\right\}$$

then, when $n=k+1$, we have

$$\begin{aligned}
\overset{k+1}{\underset{i=1}{\oplus}}\left(\omega_i\tilde{h}_i\right)&=\left(\overset{k}{\underset{i=1}{\oplus}}\left(\omega_i\tilde{h}_i\right)\right)\oplus\left(\omega_{k+1}\tilde{h}_{k+1}\right)\\
&=\left(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k}\left\{\left(f^{-1}\left(\sum_{i=1}^{k}\omega_i f\left(\gamma_i\right)\right),\prod_{i=1}^{k}w_{i\gamma_i}\right)\right\}\right)\oplus\left(\bigcup_{\gamma_{k+1}\in h_{k+1}}\left\{\left(f^{-1}\left(\omega_{k+1}f\left(\gamma_{k+1}\right)\right),w_{k+1\gamma_{k+1}}\right)\right\}\right)\\
&=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k,\gamma_{k+1}\in h_{k+1}}\left\{\left(f^{-1}\left(\sum_{i=1}^{k}\omega_i f\left(\gamma_i\right)+\omega_{k+1}f\left(\gamma_{k+1}\right)\right),\prod_{i=1}^{k+1}w_{i\gamma_i}\right)\right\}\\
&=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k,\gamma_{k+1}\in h_{k+1}}\left\{\left(f^{-1}\left(\sum_{i=1}^{k+1}\omega_i f\left(\gamma_i\right)\right),\prod_{i=1}^{k+1}w_{i\gamma_i}\right)\right\}
\end{aligned}$$

i.e., Eq. (14) holds for $n=k+1$. Thus Eq. (14) holds for all n.

In addition, because $g:[0,1]\to[0,+\infty]$ is a strictly decreasing function and $f\left(t\right)=g\left(1-t\right)$, $f:[0,1]\to[0,+\infty]$ is a strictly increasing function, which implies that

$$0\le f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\gamma_i\right)\right)\le 1$$

Furthermore, we have

$$\begin{aligned}
\sum_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left(\prod_{i=1}^{n}w_{i\gamma_i}\right)&=\sum_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_{n-1}\in h_{n-1}}\left(\prod_{i=1}^{n-1}\left(w_{i\gamma_i}\cdot\left(\sum_{\gamma_n\in h_n}w_{n\gamma_n}\right)\right)\right)\\
&=\sum_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_{n-1}\in h_{n-1}}\left(\prod_{i=1}^{n-1}w_{i\gamma_i}\right)=\cdots=\sum_{\gamma_1\in h_1}\left(w_{1\gamma_1}\cdot\left(\sum_{\gamma_2\in h_2}w_{2\gamma_2}\right)\right)\\
&=\sum_{\gamma_1\in h_1}\left(w_{1\gamma_1}\right)=1
\end{aligned}$$

This completes the proof of Theorem 4.1.
In the following, let’s study some desirable properties of the ATS-WHFWA operator.

Theorem 4.2. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of WHFEs, $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ be their weight vector with $\omega_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}\omega_i=1$, if $r>0$, then

$$\text{ATS-WHFWA}\left(r\tilde{h}_1, r\tilde{h}_2, \cdots, r\tilde{h}_n\right) = r\text{ATS-WHFWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) \tag{15}$$

Proof. Since for any $i = 1, 2, \cdots, n$,

$$r\tilde{h}_i = \bigcup_{\gamma_i \in h_i} \left\{ \left(f^{-1}\left(rf\left(\gamma_i\right)\right), w_{i\gamma_i} \right) \right\}$$

Based on Theorem 4.1, we have

$$\begin{aligned}
&\text{ATS-WHFWA}\left(r\tilde{h}_1, r\tilde{h}_2, \cdots, r\tilde{h}_n\right) \\
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(f^{-1}\left(rf\left(\gamma_i\right)\right)\right)\right), \prod_{i=1}^{n} w_{i\gamma_i} \right) \right\} \\
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \left(f^{-1}\left(r\sum_{i=1}^{n} \omega_i f\left(\gamma_i\right)\right), \prod_{i=1}^{n} w_{i\gamma_i} \right) \right\}
\end{aligned}$$

According to Eq. (11), we can get

$$\begin{aligned}
&r\text{ATS-WHFWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) \\
&= r\left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(\gamma_i\right)\right), \prod_{i=1}^{n} w_{i\gamma_i} \right) \right\} \right) \\
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \left(f^{-1}\left(rf\left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(\gamma_i\right)\right)\right)\right), \prod_{i=1}^{n} w_{i\gamma_i} \right) \right\} \\
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \left(f^{-1}\left(r\sum_{i=1}^{n} \omega_i f\left(\gamma_i\right)\right), \prod_{i=1}^{n} w_{i\gamma_i} \right) \right\}
\end{aligned}$$

This completes the proof of Theorem 4.2. □ □

Theorem 4.3. Let $\tilde{h}_i = \bigcup_{\gamma_i \in h_i} \left\{ \left(\gamma_i, w_{i\gamma_i} \right) \right\}$ and $\tilde{l}_i = \bigcup_{\xi_i \in l_i} \left\{ \left(\xi_i, \varpi_{i\xi_i} \right) \right\}$ ($i = 1, 2, \cdots, n$) be two collections of WHFEs, $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n\right)^T$ be their weight vector with $\omega_i \in [0,1]$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} \omega_i = 1$, then

$$\begin{aligned}
&\text{ATS-WHFWA}\left(\tilde{h}_1 \oplus \tilde{l}_1, \tilde{h}_2 \oplus \tilde{l}_2, \cdots, \tilde{h}_n \oplus \tilde{l}_n\right) \\
&= \text{ATS-WHFWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) \oplus \text{ATS-WHFWA}\left(\tilde{l}_1, \tilde{l}_2, \cdots, \tilde{l}_n\right)
\end{aligned} \tag{16}$$

Proof. According to Eq. (9), we have

$$\tilde{h}_i \oplus \tilde{l}_i = \bigcup_{\gamma_i \in h_i, \xi_i \in l_i} \left\{ \left(f^{-1}\left(f(\gamma_i) + f(\xi_i) \right), w_{i\gamma_i} \cdot \varpi_{i\xi_i} \right) \right\}$$

According to Theorem 4.1, we have

$$\begin{aligned}
&\text{ATS-WHFWA}\left(\tilde{h}_1 \oplus \tilde{l}_1, \tilde{h}_2 \oplus \tilde{l}_2, \cdots, \tilde{h}_n \oplus \tilde{l}_n\right) \\
&= \bigcup_{\gamma_1 \in h_1, \cdots, \gamma_n \in h_n, \xi_1 \in l_1, \cdots, \xi_n \in l_n} \left\{ \left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(f^{-1}\left(f(\gamma_i) + f(\xi_i) \right) \right) \right), \prod_{i=1}^{n} \left(w_{i\gamma_i} \cdot \varpi_{i\xi_i} \right) \right) \right\} \\
&= \bigcup_{\gamma_1 \in h_1, \cdots, \gamma_n \in h_n, \xi_1 \in l_1, \cdots, \xi_n \in l_n} \left\{ \left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i) + \sum_{i=1}^{n} \omega_i f(\xi_i) \right), \prod_{i=1}^{n} \left(w_{i\gamma_i} \cdot \varpi_{i\xi_i} \right) \right) \right\}
\end{aligned}$$

On other hand, according to Theorem 4.1 and Eq. (9), we have

$$\begin{aligned}
&\text{ATS-WHFWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) \oplus \text{ATS-WHFWA}\left(\tilde{l}_1, \tilde{l}_2, \cdots, \tilde{l}_n\right) \\
&= \left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i) \right), \prod_{i=1}^{n} w_{i\gamma_i} \right) \right\} \right) \oplus \left(\bigcup_{\xi_1 \in l_1, \xi_2 \in l_2, \cdots, \xi_n \in l_n} \left\{ \left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\xi_i) \right), \prod_{i=1}^{n} \varpi_{i\xi_i} \right) \right\} \right) \\
&= \bigcup_{\gamma_1 \in h_1, \cdots, \gamma_n \in h_n, \xi_1 \in l_1, \cdots, \xi_n \in l_n} \left\{ \left(f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i) \right) \right) + f\left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\xi_i) \right) \right) \right), \prod_{i=1}^{n} w_{i\gamma_i} \cdot \prod_{i=1}^{n} \varpi_{i\xi_i} \right) \right\} \\
&= \bigcup_{\gamma_1 \in h_1, \cdots, \gamma_n \in h_n, \xi_1 \in l_1, \cdots, \xi_n \in l_n} \left\{ \left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i) + \sum_{i=1}^{n} \omega_i f(\xi_i) \right), \prod_{i=1}^{n} \left(w_{i\gamma_i} \cdot \varpi_{i\xi_i} \right) \right) \right\}
\end{aligned}$$

which completes the proof of Theorem 4.3. □ □

If the additive generator g is assigned different forms, then some specific weighted hesitant fuzzy aggregation operators can be obtained as follows:

Case 1. If $g(t) = -\log(t)$, then the ATS-WHFWA operator reduces to the following form:

$$\text{WHFWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \left(1 - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}, \prod_{i=1}^{n} w_{i\gamma_i} \right) \right\} \tag{17}$$

which is the weighted hesitant fuzzy weighted averaging (WHFWA) operator.

In fact, if $g(t) = -\log(t)$, then $f(t) = g(1-t) = -\log(1-t)$ and $f^{-1}(t) = 1 - e^{-t}$. Thus,

$$\begin{aligned}\text{WHFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right) &= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\gamma_i\right)\right),\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \\ &= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-e^{\sum_{i=1}^{n}\omega_i\log\left(1-\gamma_i\right)},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \\ &= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-e^{\log\left(\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}\right)},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \\ &= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\}\end{aligned}$$

Furthermore, if $w_{i\gamma_i}=\frac{1}{\#h_i}$ for any $i=1,2,\cdots,n$, where $\#h_i$ is the number of the elements in h_i, then the Eq. (17) is transformed to

$$\text{HFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i},\frac{1}{\prod_{i=1}^{n}\#h_i}\right)\right\} \tag{18}$$

which is the hesitant fuzzy weighted average (HFWA) operator proposed by Xia and Xu [15].

Case 2. If $g\left(t\right)=\log\left(\frac{2-t}{t}\right)$, then the ATS-WHFWA operator reduces to the following form:

$$\text{WHFEWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{\prod_{i=1}^{n}\left(1+\gamma_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\gamma_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \tag{19}$$

which is the weighted hesitant fuzzy Einstein weighted averaging (WHFEWA) operator.

In fact, if $g\left(t\right)=\log\left(\frac{2-t}{t}\right)$, then $f\left(t\right)=g\left(1-t\right)=\log\left(\frac{1+t}{1-t}\right)$ and $f^{-1}\left(t\right)=\frac{e^t-1}{e^t+1}$. Thus,

$$\mathrm{WHFEWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\gamma_i\right)\right),\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\}$$

$$=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{e^{\sum_{i=1}^{n}\omega_i\log\left(\frac{1+\gamma_i}{1-\gamma_i}\right)}-1}{e^{\sum_{i=1}^{n}\omega_i\log\left(\frac{1+\gamma_i}{1-\gamma_i}\right)}+1},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\}$$

$$=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+\gamma_i}{1-\gamma_i}\right)^{\omega_i}\right)}-1}{e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+\gamma_i}{1-\gamma_i}\right)^{\omega_i}\right)}+1},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\}$$

$$=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{\prod_{i=1}^{n}\left(\frac{1+\gamma_i}{1-\gamma_i}\right)^{\omega_i}-1}{\prod_{i=1}^{n}\left(\frac{1+\gamma_i}{1-\gamma_i}\right)^{\omega_i}+1},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\}$$

$$=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{\prod_{i=1}^{n}\left(1+\gamma_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\gamma_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\}$$

Furthermore, if $w_{i\gamma_i}=\dfrac{1}{\#h_i}$ for any $i=1,2,\cdots,n$, then the Eq. (19) is transformed to

$$\mathrm{HFEWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{\prod_{i=1}^{n}\left(1+\gamma_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\gamma_i\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}},\frac{1}{\prod_{i=1}^{n}\#h_i}\right)\right\} \quad (20)$$

which is the hesitant fuzzy Einstein weighted average (HFEWA) operator given by Wei and Zhao [24].

Case 3. If $g\left(t\right)=\log\left(\dfrac{\theta+\left(1-\theta\right)t}{t}\right)$, $\theta>0$, then the ATS-WHFWA operator reduces to the following form:

$$\mathrm{WHFHWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{\prod_{i=1}^{n}\left(1+\left(\theta-1\right)\gamma_i\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\left(\theta-1\right)\gamma_i\right)^{\omega_i}+\left(\theta-1\right)\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \quad (21)$$

which is the weighted hesitant fuzzy Hammer weighted averaging (WHFHWA) operator. Especially, if $\theta = 1$, then the WHFHWA operator reduces to the WHFWA operator; if $\theta = 2$, then the WHFHWA operator reduces to the WHFEWA operator.

In fact, if $g(t) = \log\left(\frac{\theta + (1-\theta)t}{t}\right)$, then $f(t) = g(1-t) = \log\left(\frac{1+(\theta-1)t}{1-t}\right)$ and $f^{-1}(t) = \frac{1-e^t}{1-\theta-e^t}$. Thus,

$$\begin{aligned}
\mathrm{WHFHWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) &= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{\left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i)\right), \prod_{i=1}^{n} w_{i\gamma_i}\right)\right\} \\
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{\left(\frac{1 - e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+(\theta-1)\gamma_i}{1-\gamma_i}\right)^{\omega_i}\right)}}{1-\theta-e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+(\theta-1)\gamma_i}{1-\gamma_i}\right)^{\omega_i}\right)}}, \prod_{i=1}^{n} w_{i\gamma_i}\right)\right\} \\
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{\left(\frac{\prod_{i=1}^{n}\left(1+(\theta-1)\gamma_i\right)^{\omega_i} - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)\gamma_i\right)^{\omega_i} + (\theta-1)\prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}, \prod_{i=1}^{n} w_{i\gamma_i}\right)\right\}
\end{aligned}$$

Furthermore, if $w_{i\gamma_i} = \frac{1}{\#h_i}$ for any $i = 1, 2, \cdots, n$, then the Eq. (21) is transformed to

$$\mathrm{HFHWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{\left(\frac{\prod_{i=1}^{n}\left(1+(\theta-1)\gamma_i\right)^{\omega_i} - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)\gamma_i\right)^{\omega_i} + (\theta-1)\prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}, \frac{1}{\prod_{i=1}^{n} \#h_i}\right)\right\} \tag{22}$$

which is the hesitant fuzzy Hammer weighted average (HFHWA) operator. Especially, if $\theta = 1$, then the HFHWA operator reduces to the HFWA operator; if $\theta = 2$, then the HFHWA operator reduces to the HFEWA operator.

Case 4. If $g(t) = \log\left(\frac{\theta - 1}{\theta^t - 1}\right)$, $\theta > 0$, then the ATS-WHFWA operator reduces to the following form:

$$\mathrm{WHFFWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{\left(1 - \log_\theta\left(1 + \prod_{i=1}^{n}\left(\theta^{1-\gamma_i} - 1\right)^{\omega_i}\right), \prod_{i=1}^{n} w_{i\gamma_i}\right)\right\} \tag{23}$$

which is the weighted hesitant fuzzy Frank weighted averaging (WHFFWA) operator. Especially, if $\theta \to 1$, then the WHFFWA operator reduces to the WHFWA operator.

In fact, if $g(t)=\log\left(\frac{\theta-1}{\theta^t-1}\right)$, then $f(t)=g(1-t)=\log\left(\frac{\theta-1}{\theta^{1-t}-1}\right)$ and $f^{-1}(t)=1-\log_\theta\left(\frac{\theta-1+e^t}{e^t}\right)$. Thus,

$$\begin{aligned}
\text{WHFFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right) &= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(f^{-1}\left(\sum_{i=1}^n \omega_i f(\gamma_i)\right),\prod_{i=1}^n w_{i\gamma_i}\right)\right\} \\
&= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-\log_\theta\left(\frac{\theta-1+e^{\log\left(\prod_{i=1}^n\left(\frac{\theta-1}{\theta^{1-\gamma_i}-1}\right)^{\omega_i}\right)}}{e^{\log\left(\prod_{i=1}^n\left(\frac{\theta-1}{\theta^{1-\gamma_i}-1}\right)^{\omega_i}\right)}}\right),\prod_{i=1}^n w_{i\gamma_i}\right)\right\} \\
&= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-\log_\theta\left(\frac{\theta-1+\prod_{i=1}^n\left(\frac{\theta-1}{\theta^{1-\gamma_i}-1}\right)^{\omega_i}}{\prod_{i=1}^n\left(\frac{\theta-1}{\theta^{1-\gamma_i}-1}\right)^{\omega_i}}\right),\prod_{i=1}^n w_{i\gamma_i}\right)\right\} \\
&= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-\log_\theta\left(1+\prod_{i=1}^n\left(\theta^{1-\gamma_i}-1\right)^{\omega_i}\right),\prod_{i=1}^n w_{i\gamma_i}\right)\right\}
\end{aligned}$$

Furthermore, if $w_{i\gamma_i}=\frac{1}{\#h_i}$ for any $i=1,2,\cdots,n$, then the Eq. (23) is transformed to

$$\text{HFFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-\log_\theta\left(1+\prod_{i=1}^n\left(\theta^{1-\gamma_i}-1\right)^{\omega_i}\right),\frac{1}{\prod_{i=1}^n \#h_i}\right)\right\} \quad (24)$$

which is the hesitant fuzzy Frank weighted average (HFFWA) operator.
Based on the ATS-WHFWA operator and the geometric mean, here we define an Archimedean t-conorm and t-norm based weighted hesitant fuzzy weighted geometric (ATS-WHFWG) operator:

Definition 4.2. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of WHFEs, and let $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ be the weight vector of $\tilde{h}_i$ ($i=1,2,\cdots,n$) with $\omega_i\in[0,1]$ and $\sum_{i=1}^n\omega_i=1$. Then, an Archimedean t-conorm and t-norm based weighted hesitant fuzzy weighted geometric (ATS-WHFWG) operator is a mapping $\tilde{H}^n\to\tilde{H}$, where

$$\text{ATS-WHFWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigotimes_{i=1}^n\left(\tilde{h}_i^{\omega_i}\right) \quad (25)$$

Theorem 4.4. Let $\tilde{h}_i = \bigcup_{\gamma_i \in h_i} \left\{ \left(\gamma_i, w_{i\gamma_i} \right) \right\}$ ($i = 1, 2, \cdots, n$) be a collection of WHFEs, and $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n \right)^T$ be the weight vector of $\tilde{h}_i$ ($i = 1, 2, \cdots, n$), where ω_i indicates the importance degree of $\tilde{h}_i$, satisfying $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$, then the aggregated value by using the ATS-WHFWG operator is also a WHFE, and

$$\text{ATS-WHFWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n \right) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \left(g^{-1}\left(\sum_{i=1}^{n} \omega_i g\left(\gamma_i \right) \right), \prod_{i=1}^{n} w_{i\gamma_i} \right) \right\} \tag{26}$$

Theorem 4.5. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of WHFEs, $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n \right)^T$ be their weight vector with $\omega_i \in [0,1]$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} \omega_i = 1$, if $r > 0$, then

$$\text{ATS-WHFWG}\left(\tilde{h}_1^r, \tilde{h}_2^r, \cdots, \tilde{h}_n^r \right) = \left(\text{ATS-WHFWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n \right) \right)^r \tag{27}$$

Theorem 4.6. Let $\tilde{h}_i = \bigcup_{\gamma_i \in h_i} \left\{ \left(\gamma_i, w_{i\gamma_i} \right) \right\}$ and $\tilde{l}_i = \bigcup_{\xi_i \in l_i} \left\{ \left(\xi_i, \varpi_{i\xi_i} \right) \right\}$ ($i = 1, 2, \cdots, n$) be two collections of WHFEs, $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n \right)^T$ be their weight vector with $\omega_i \in [0,1]$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} \omega_i = 1$, then

$$\begin{aligned} &\text{ATS-WHFWG}\left(\tilde{h}_1 \otimes \tilde{l}_1, \tilde{h}_2 \otimes \tilde{l}_2, \cdots, \tilde{h}_n \otimes \tilde{l}_n \right) \\ &= \text{ATS-WHFWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n \right) \otimes \text{ATS-WHFWG}\left(\tilde{l}_1, \tilde{l}_2, \cdots, \tilde{l}_n \right) \end{aligned} \tag{28}$$

In what follows, we will investigate the relationship between ATS-WHFWA operator and ATS-WHFWG operator.

Theorem 4.7. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of WHFEs, $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n \right)^T$ be their weight vector with $\omega_i \in [0,1]$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} \omega_i = 1$, then we have

$$\text{ATS-WHFWA}\left(\tilde{h}_1^c, \tilde{h}_2^c, \cdots, \tilde{h}_n^c \right) = \left(\text{ATS-WHFWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n \right) \right)^c \tag{29}$$

$$\text{ATS-WHFWG}\left(\tilde{h}_1^c, \tilde{h}_2^c, \cdots, \tilde{h}_n^c \right) = \left(\text{ATS-WHFWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n \right) \right)^c \tag{30}$$

Proof. (1) According to Eqs. (6), (14), and (26), we can get

$$\begin{aligned}
\text{ATS-WHFWA}\left(\tilde{h}_1^c,\tilde{h}_2^c,\cdots,\tilde{h}_n^c\right) &= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(1-\gamma_i\right)\right),\prod_{i=1}^{n} w_{i\gamma_i}\right)\right\} \\
&= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\gamma_i\right)\right),\prod_{i=1}^{n} w_{i\gamma_i}\right)\right\} \\
&= \left(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\gamma_i\right)\right),\prod_{i=1}^{n} w_{i\gamma_i}\right)\right\}\right)^c \\
&= \left(\text{ATS-WHFWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\right)^c
\end{aligned}$$

(2) According to Eqs. (6), (14), and (26), we have

$$\begin{aligned}
\text{ATS-WHFWG}\left(\tilde{h}_1^c,\tilde{h}_2^c,\cdots,\tilde{h}_n^c\right) &= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(1-\gamma_i\right)\right),\prod_{i=1}^{n} w_{i\gamma_i}\right)\right\} \\
&= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\gamma_i\right)\right),\prod_{i=1}^{n} w_{i\gamma_i}\right)\right\} \\
&= \left(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\gamma_i\right)\right),\prod_{i=1}^{n} w_{i\gamma_i}\right)\right\}\right)^c \\
&= \left(\text{ATS-WHFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\right)^c
\end{aligned}$$

This completes the proof of Theorem 4.7.

Theorem 4.8. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of WHFEs, $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ be their weight vector with $\omega_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}\omega_i=1$, if $\tilde{h}_i=\tilde{h}$ for all i, then

$$\text{ATS-WHFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\tilde{h}$$

Theorem 4.9. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of WHFEs, $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ be their weight vector with $\omega_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}\omega_i=1$, if $\tilde{h}$ is a WHFE, then

$$\text{ATS-WHFWA}\left(\tilde{h}_1\oplus\tilde{h},\tilde{h}_2\oplus\tilde{h},\cdots,\tilde{h}_n\oplus\tilde{h}\right)=\text{ATS-WHFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\oplus\tilde{h}$$

Theorem 4.10. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of WHFEs, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be their weight vector with $\omega_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}\omega_i=1$, if $r>0$ and $\tilde{h}$ is a WHFE, then

$$\text{ATS-WHFWA}\left(r\tilde{h}_1\oplus\tilde{h},r\tilde{h}_2\oplus\tilde{h},\cdots,r\tilde{h}_n\oplus\tilde{h}\right)=r\text{ATS-WHFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\oplus\tilde{h}$$

Theorem 4.11. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of WHFEs, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be their weight vector with $\omega_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}\omega_i=1$, if $\tilde{h}_i=\tilde{h}$ for all i, then

$$\text{ATS-WHFWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\tilde{h}$$

Theorem 4.12. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of WHFEs, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be their weight vector with $\omega_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}\omega_i=1$, if $\tilde{h}$ is a WHFE, then

$$\text{ATS-WHFWG}\left(\tilde{h}_1\otimes\tilde{h},\tilde{h}_2\otimes\tilde{h},\cdots,\tilde{h}_n\otimes\tilde{h}\right)=\text{ATS-WHFWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\otimes\tilde{h}$$

Theorem 4.13. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of WHFEs, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be their weight vector with $\omega_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}\omega_i=1$, if $r>0$ and $\tilde{h}$ is a WHFE, then

$$\text{ATS-WHFWA}\left(\tilde{h}_1^r\otimes\tilde{h},\tilde{h}_2^r\otimes\tilde{h},\cdots,\tilde{h}_n^r\otimes\tilde{h}\right)=\left(\text{ATS-WHFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\right)^r\otimes\tilde{h}$$

If the additive generator g is assigned different forms, then some specific ATS-WHFWG operators can be obtained as follows:

Case 1. If $g(t)=-\log(t)$, then the ATS-WHFWG operator reduces to the following form:

$$\text{WHFWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\prod_{i=1}^{n}\gamma_i^{\omega_i},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \tag{31}$$

which is the weighted hesitant fuzzy weighted geometric (WHFWG) operator. Furthermore, if $w_{i\gamma_i}=\dfrac{1}{\#h_i}$ for any $i=1,2,\cdots,n$, where $\#h_i$ is the number of the elements in h_i, then the Eq. (31) is transformed to

$$\mathrm{HFWG}_{\lambda}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\prod_{i=1}^{n}\gamma_i^{\omega_i},\frac{1}{\prod_{i=1}^{n}\#h_i}\right)\right\} \tag{32}$$

which is the hesitant fuzzy weighted geometric (HFWG) operator proposed by Xia and Xu [15].

Case 2. If $g(t)=\log\left(\frac{2-t}{t}\right)$, then the ATS-WHFWG operator reduces to the following form:

$$\mathrm{WHFEWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{2\prod_{i=1}^{n}\gamma_i^{\omega_i}}{\prod_{i=1}^{n}\left(2-\gamma_i\right)^{\omega_i}+\prod_{i=1}^{n}\gamma_i^{\omega_i}},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \tag{33}$$

which is the weighted hesitant fuzzy Einstein weighted geometric (WHFEWG) operator. Furthermore, if $w_{i\gamma_i}=\frac{1}{\#h_i}$ for any $i=1,2,\cdots,n$, then the Eq. (33) is transformed to

$$\mathrm{HFEWG}_{\lambda}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{2\prod_{i=1}^{n}\gamma_i^{\omega_i}}{\prod_{i=1}^{n}\left(2-\gamma_i\right)^{\omega_i}+\prod_{i=1}^{n}\gamma_i^{\omega_i}},\frac{1}{\prod_{i=1}^{n}\#h_i}\right)\right\} \tag{34}$$

which is the hesitant fuzzy Einstein weighted geometric (HFEWG) operator given by Wei and Zhao [24].

Case 3. If $g(t)=\log\left(\frac{\theta+(1-\theta)t}{t}\right)$, $\theta>0$, then the ATS-WHFWG operator reduces to the following form:

$$\mathrm{WHFHWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{\theta\prod_{i=1}^{n}\gamma_i^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)(1-\gamma_i)\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\gamma_i^{\omega_i}},\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \tag{35}$$

which is the weighted hesitant fuzzy Hammer weighted geometric (WHFHWG) operator. Especially, if $\theta=1$, then the WHFHWG operator reduces to the WHFWG operator; if $\theta=2$, then the WHFHWG operator reduces to the WHFEWG operator.

Furthermore, if $w_{i\gamma_i}=\frac{1}{\#h_i}$ for any $i=1,2,\cdots,n$, then the Eq. (35) is transformed to

$$\text{HFHWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\frac{\theta\prod_{i=1}^{n}\gamma_i^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)(1-\gamma_i)\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\gamma_i^{\omega_i}},\frac{1}{\prod_{i=1}^{n}\#h_i}\right)\right\} \quad (36)$$

which is the hesitant fuzzy Hammer weighted geometric (HFHWG) operator. Especially, if $\theta=1$, then the HFHWG operator reduces to the HFWG operator; if $\theta=2$, then the HFHWG operator reduces to the HFEWG operator.

Case 4. If $g(t)=\log\left(\frac{\theta-1}{\theta^t-1}\right)$, $\theta>0$, then the ATS-WHFWG operator reduces to the following form:

$$\text{WHFFWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\gamma_i}-1\right)^{\omega_i}\right),\prod_{i=1}^{n}w_{i\gamma_i}\right)\right\} \quad (37)$$

which is the weighted hesitant fuzzy Frank weighted geometric (WHFFWG) operator.

Furthermore, if $w_{i\gamma_i}=\frac{1}{\#h_i}$ for any $i=1,2,\cdots,n$, then the Eq. (37) is transformed to

$$\text{HFFWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\gamma_i}-1\right)^{\omega_i}\right),\frac{1}{\prod_{i=1}^{n}\#h_i}\right)\right\} \quad (38)$$

which is the hesitant fuzzy Frank weighted geometric (HFFWG) operator.

5 An Approach to Multi-Criteria Decision Making with Weighted Hesitant Fuzzy Information

In this section, we shall utilize the proposed operators to develop an approach to multi-criteria decision making (MCDM) with weighted hesitant fuzzy information. For a MCDM problem, let $Y=\{Y_1,Y_2,\cdots,Y_m\}$ be a set of m alternatives, $G=\{G_1,G_2,\ldots,G_n\}$ be a collection of n criteria, whose weight vector is $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$, with $\omega_j\in[0,1]$, $j=1,2,\cdots,n$, and $\sum_{j=1}^{n}\omega_j=1$, where ω_j denotes the importance degree of the criterion G_j. The decision makers provide all the possible values with their corresponding weights for the alternative Y_i with respect to the criterion G_j represented by a WHFE $\tilde{r}_{ij} = \bigcup_{\gamma_{ij} \in r_{ij}} \left\{ \left(\gamma_{ij}, w_{ij\gamma_{ij}} \right) \right\}$. All $\tilde{r}_{ij}$ ($i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$) construct the weighted hesitant fuzzy decision matrix $R = \left(\tilde{r}_{ij} \right)_{m \times n}$ (see Table 1).

Table 1. The weighted hesitant fuzzy decision matrix R

	G_1	$\cdots$	G_j	$\cdots$	G_n
Y_1	$\tilde{r}_{11}$	$\cdots$	$\tilde{r}_{1j}$	$\cdots$	$\tilde{r}_{1n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Y_i	$\tilde{r}_{i1}$	$\cdots$	$\tilde{r}_{ij}$	$\cdots$	$\tilde{r}_{in}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Y_m	$\tilde{r}_{m1}$	$\cdots$	$\tilde{r}_{mj}$	$\cdots$	$\tilde{r}_{mn}$

In general, there are benefit attributes (i.e., the bigger the attribute values the better) and cost attributes (i.e., the smaller the attribute values the better) in a MCDM problem. In such cases, we need transform the attribute values of cost type into the attribute values of benefit type, i.e., transform the weighted hesitant fuzzy decision matrix $R = \left(\tilde{r}_{ij} \right)_{m \times n}$ into a normalized weighted hesitant fuzzy decision matrix $A = \left(\tilde{a}_{ij} \right)_{m \times n}$ by the method given by Xu and Hu [30], where

$$\tilde{a}_{ij} = \begin{cases} \tilde{r}_{ij}, & \text{for benefit attribute } G_j \\ \tilde{r}_{ij}^{c}, & \text{for cost attribute } G_j \end{cases}, \quad i = 1, 2, \cdots, m, \ j = 1, 2, \cdots, n, \tag{39}$$

where $\tilde{r}_{ij}^{c}$ is the complement of $\tilde{r}_{ij}$ such that $\tilde{r}_{ij}^{c} = \bigcup_{\gamma_{ij} \in r_{ij}} \left\{ \left(1 - \gamma_{ij}, w_{ij\gamma_{ij}} \right) \right\}$.

Step 1. Transform the weighted hesitant fuzzy decision matrix $R = \left(\tilde{r}_{ij} \right)_{m \times n}$ into the normalized weighted hesitant fuzzy decision matrix $A = \left(\tilde{a}_{ij} \right)_{m \times n}$ based on Eq. (39).

Step 2. Utilize the ATS-WHFWA operator (Eq. (14))

$$\tilde{a}_i = \text{ATS-WHFWA}\left(\tilde{a}_{i1}, \tilde{a}_{i2}, \cdots, \tilde{a}_{in} \right) = \bigcup_{\gamma_{i1} \in a_{i1}, \gamma_{i2} \in a_{i2}, \cdots, \gamma_{in} \in a_{in}} \left\{ \left(f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left(\gamma_{ij} \right) \right), \prod_{j=1}^{n} w_{ij\gamma_{ij}} \right) \right\} \tag{40}$$

or the ATS-WHFWG operator (Eq. (26))

$$\tilde{a}_i = \text{ATS-WHFWG}\left(\tilde{a}_{i1}, \tilde{a}_{i2}, \cdots, \tilde{a}_{in} \right) = \bigcup_{\gamma_{i1} \in a_{i1}, \gamma_{i2} \in a_{i2}, \cdots, \gamma_{in} \in a_{in}} \left\{ \left(g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\gamma_{ij} \right) \right), \prod_{j=1}^{n} w_{ij\gamma_{ij}} \right) \right\} \tag{41}$$

to aggregate all the performance values $\tilde{a}_{ij}$ ($j = 1, 2, \cdots, n$) in the ith line of A, and then derive the overall performance value $\tilde{a}_i$ ($i = 1, 2, \cdots, m$) of the alternative Y_i ($i = 1, 2, \cdots, m$).

Step 3. According to Definition 3.2, calculate the scores $s(\tilde{a}_i)$ ($i=1,2,\cdots,m$) of $\tilde{a}_i$ ($i=1,2,\cdots,m$) and rank all the alternatives Y_i ($i=1,2,\cdots,m$) according to $s(\tilde{a}_i)$ in descending order.

Step 4. End.

In the following, we use a numerical example adapted from [20,22] to illustrate our approach.

Example 5.1 [20,22]. Suppose that a factory intends to select a new site for new buildings. Assume that there are three possible alternatives Y_i ($i=1,2,3$) and three criteria are considered to decide which site to choose: (1) G_1 (price); (2) G_2 (location); and (3) G_3 (environment). The weight vector of three criteria G_j $(j=1,2,3)$ is $\omega=(0.3,0.2,0.5)^T$. Suppose that the characteristics of the alternatives Y_i ($i=1,2,3$) with respect to the criteria G_j ($j=1,2,3$) are denoted by the WHFE $\tilde{r}_{ij}=\bigcup_{\gamma_{ij}\in r_{ij}}\left\{\left(\gamma_{ij},w_{ij\gamma_{ij}}\right)\right\}$, where γ_{ij} indicates the possible degree to which the alternative Y_i satisfies the criterion G_j and $w_{ij\gamma_{ij}}$ is the weight of γ_{ij}. All $\tilde{r}_{ij}$ ($i=1,2,3$; $j=1,2,3$) are contained in the weighted hesitant fuzzy decision matrix $R=\left(\tilde{r}_{ij}\right)_{m\times n}$ (see Table 2).

Table 2. The weighted hesitant fuzzy decision matrix R

	G_1	G_2	G_3
Y_1	{(0.6, 0.3), (0.5, 0.3), (0.4, 0.4)}	{(0.6, 0.8), (0.4, 0.2)}	{(0.5, 0.3), (0.3, 0.7)}
Y_2	{(0.4, 0.6), (0.3, 0.4)}	{(0.8, 1)}	{(0.4, 0.2), (0.3, 0.3), (0.2, 0.5)}
Y_3	{(0.8, 1)}	{(0.7, 0.1), (0.6, 0.3), (0.5, 0.6)}	{(0.2, 0.5), (0.1, 0.5)}

Step 1. Because all of the criteria G_j ($j=1,2,3$) are of the benefit type, the performance values of the alternatives Y_i ($i=1,2,3$) do not require normalization.

Step 2. Utilize the WHFHWA operator (Eq. (21)) (suppose that $\theta=1$) to aggregate all the preference values $\tilde{r}_{ij}$ ($j=1,2,3$) in the ith line of R, and then derive the overall performance value $\tilde{r}_i$ ($i=1,2,3$) of the alternative Y_i ($i=1,2,3$):

$$\tilde{r}_1=\left\{\begin{array}{l}(0.5528,0.0720),(0.4708,0.1680),(0.5150,0.0180),(0.4262,0.0420),\\(0.5218,0.0720),(0.4342,0.1680),(0.4814,0.0180),(0.3864,0.0420),\\(0.4949,0.0960),(0.4024,0.2240),(0.4523,0.0240),(0.3519,0.0560)\end{array}\right\}$$

$$\tilde{r}_2 = \begin{Bmatrix} (0.5184,0.1200),(0.4798,0.1800),(0.4438,0.3000), \\ (0.4956,0.0800),(0.4551,0.1200),(0.4175,0.2000) \end{Bmatrix}$$

$$\tilde{r}_3 = \begin{Bmatrix} (0.5662,0.0500),(0.5399,0.0500),(0.5405,0.1500), \\ (0.5126,0.1500),(0.5196,0.3000),(0.4904,0.3000) \end{Bmatrix}$$

Step 3. According to Definition 3.2, we calculate the score values $s(\tilde{r}_i)$ ($i=1,2,3$) of $\tilde{r}_i$ ($i=1,2,3$) as:

$$s(\tilde{r}_1)=0.4497, \quad s(\tilde{r}_2)=0.4595, \quad s(\tilde{r}_3)=0.5163.$$

Step 4. Since $s(\tilde{r}_3) > s(\tilde{r}_2) > s(\tilde{r}_1)$, then we get the ranking of the alternatives Y_i $(i=1,2,3)$ as $Y_3 > Y_2 > Y_1$. Thus, the best alternative is Y_3.

In the following, we will analyze how different values of the parameter θ change the aggregation results. As θ is assigned different values between 0 and 30, the score functions of the alternatives obtained by the WHFHWA operator are shown in Fig. 1.

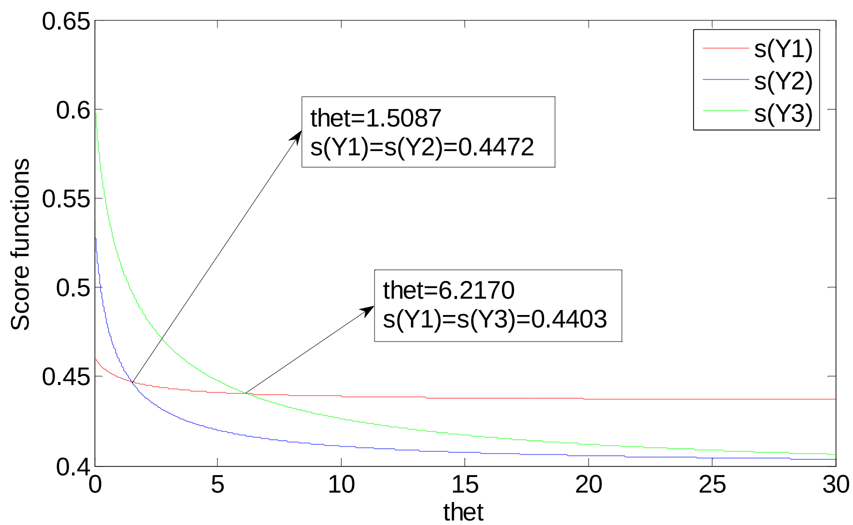

Fig. 1. Score functions for alternatives obtained by the WHFHWA operator

Fig. 1 demonstrates that all the score functions decrease as θ increases from 0 to 30, from which we can find that

(1) when $\theta \in (0,1.5087]$, the ranking of the four alternatives is $Y_3 > Y_2 > Y_1$ and the best choice is Y_3.

(2) when $\theta \in (1.5087,6.2170]$, the ranking of the four alternatives is $Y_3 > Y_1 > Y_2$ and the best choice is Y_3.

(3) when $\theta \in (6.2170,30]$, the ranking of the four alternatives is $Y_1 > Y_3 > Y_2$ and the best choice is Y_1.

In the above example, if we use the WHFHWG operator instead of the WHFHWA operator to aggregate the values of the alternatives, then the score functions of the alternatives are shown in Fig. 2. From Fig. 2, we can see that all the score functions obtained by the WHFHWG operator increase as the parameter θ increases from 0 to 30 and the aggregation arguments are kept fixed. From Fig. 2, we can also see that as θ increases from 0 to 30, the ranking of alternatives is always $Y_1 > Y_2 > Y_3$ and the best choice is always Y_1.

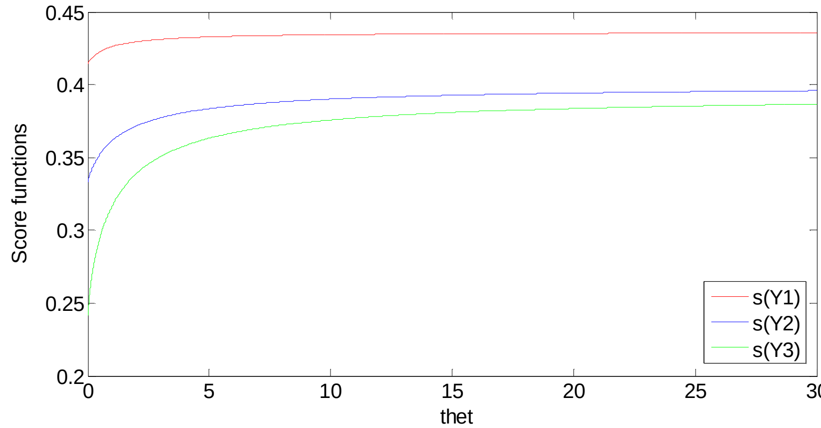

Fig. 2. Score functions for alternatives obtained by the WHFHWG operator

Fig. 3 illustrates the deviation values between the score functions obtained by the WHFHWA operator and the ones obtained by the WHFHWG operator, from which we can find that the values obtained by the WHFHWA operator are greater than the ones obtained by the WHFHWG operator for the same value of the parameter θ and the same aggregation values, and the deviation values decrease as the value of the parameter θ increases.

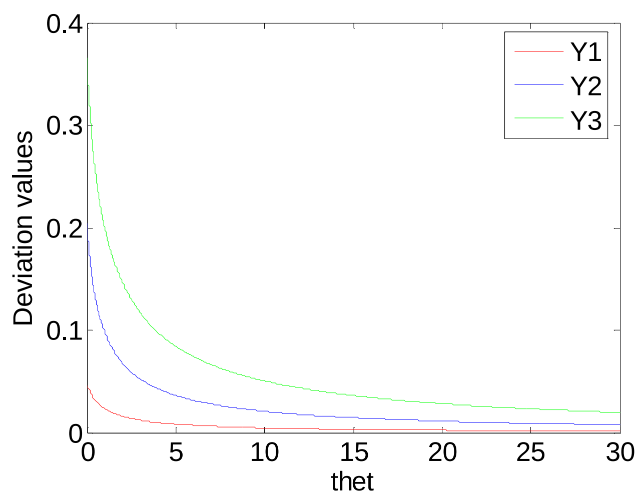

Fig. 3. Deviation values for alternatives between the WHFHWA and WHFHWG operators

Fig. 3 indicates that the WHFHWA operator can obtain more favorable (or optimistic) expectations, and therefore can be considered as an optimistic operator, while the WHFHWG operator has more unfavorable (or pessimistic) expectations, and therefore can be considered as a pessimistic operator. The values of the parameter θ can be considered as the optimistic or pessimistic levels. According to Figs. 1, 2, and 3, we can conclude that the decision makers who take a gloomy view of the prospects could use the WHFHWG operator and choose the smaller values of the parameter θ , while the decision makers who are optimistic could use the WHFHWA operator and choose the smaller values of the parameter θ .

If we use the WHFFWA (or WHFFWG) operator instead of the WHFHWA (or WHFHWG) operator to aggregate the attribute values of alternatives, then the score functions of alternatives are given in Figs. 4 and 5, respectively. Fig. 4 shows that all the score functions obtained by the WHFFWA operator decrease as the parameter θ increases from 0 to 30, from which we can get that

(1) when $\theta \in (0, 2.3659]$, the ranking of the four alternatives is $Y_3 > Y_2 > Y_1$ and the best choice is Y_3.
(2) when $\theta \in (2.3659, 30]$, the ranking of the four alternatives is $Y_3 > Y_1 > Y_2$ and the best choice is Y_3.

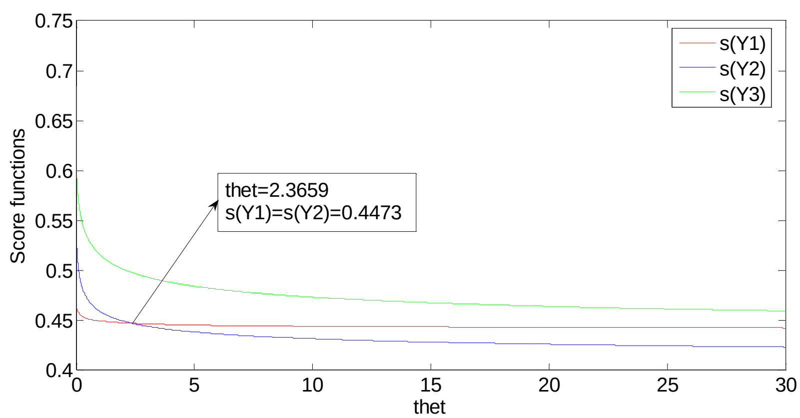

Fig. 4. Score functions for alternatives obtained by the WHFFWA operator

Fig. 5 illustrates that all the score functions obtained by the WHFFWG operator increase as the parameter θ increases from 0 to 30, from which we can see that as θ increases from 0 to 30, the ranking of alternatives is always $Y_1 > Y_2 > Y_3$ and the best choice is always Y_1.

Fig. 6 illustrates the deviation values between the score functions obtained by the WHFFWA operator and the ones obtained by the WHFFWG operator, from which we can find that the values obtained by the WHFFWA operator are greater than the ones obtained by the WHFFWG operator for the same value of the parameter θ and the same aggregation values, and the deviation values decrease as the value of the parameter θ increases.

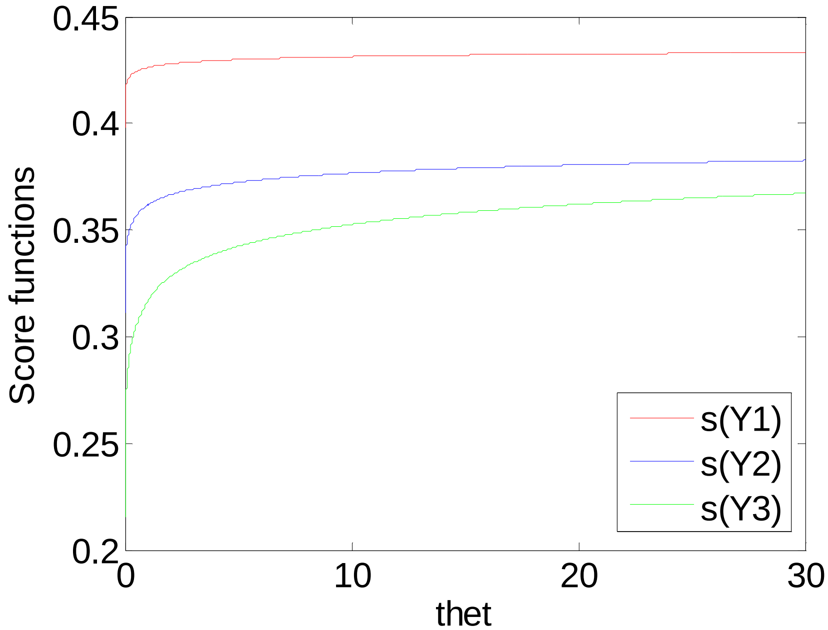

Fig. 5. Score functions for alternatives obtained by the WHFFWG operator

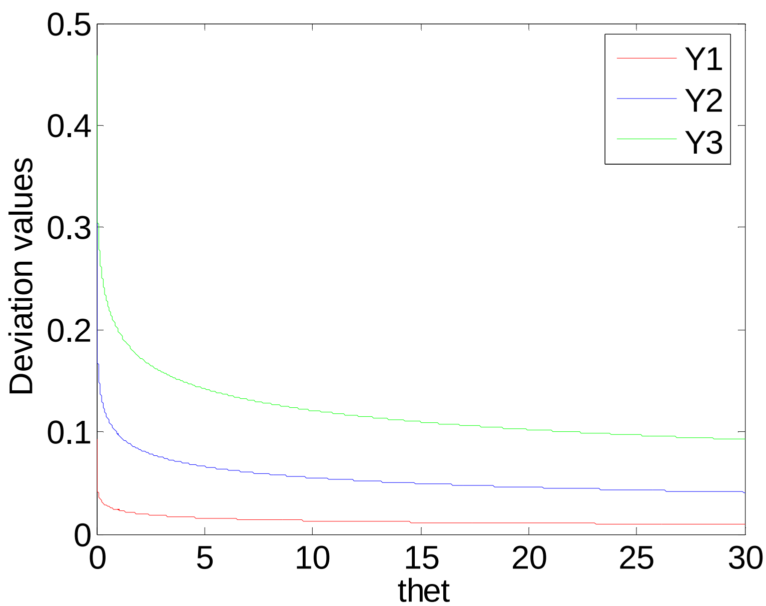

Fig. 6. Deviation values for alternatives between the WHFFWA and WHFFWG operators

Fig. 6 indicates that the WHFFWA operator can obtain more favorable (or optimistic) expectations, and therefore can be considered as an optimistic operator, while the WHFFWG operator has more unfavorable (or pessimistic) expectations, and therefore can be considered as a pessimistic operator. The values of the parameter θ can be considered as the optimistic or pessimistic levels. According to Figs. 4, 5, and 6, we can conclude that the decision makers who take a gloomy view of the prospects could use the WHFFWG operator and choose the smaller values of the parameter θ, while the decision makers who are optimistic could use the WHFFWA operator and choose the smaller values of the parameter θ.

Based on the above analysis, we can see that the parameter θ reflects the decision makers' preferences and the decision makers can choose the proper values of θ according to their preferences. By choosing different values of the parameter θ, we can derive different score functions, and then derive the different rankings of the alternatives and the different optimal alternatives. That is, the final optimal decisions based on different values of the parameter θ could be different. Therefore, the developed aggregation operators with the parameters can provide us with more choices and more flexibility than the existing ones due to the fact they allow us to choose different values of the parameter in the light of the different practical situations.

Example 5.2 (Continued with Example 5.1). In Example 5.1, if we do not consider the importance of all of the possible values for an alternative under an attribute and only use the HFEs to represent the performance values of an alternative under an attribute, then the weighted hesitant fuzzy decision matrix R reduces to the hesitant fuzzy decision matrix $R' = \left(\tilde{r}'_{ij}\right)_{3\times 3}$ (see Table 3).

Table 3. The hesitant fuzzy decision matrix R'

	G_1	G_2	G_3
Y_1	{0.6, 0.5, 0.4}	{0.6, 0.4}	{0.5, 0.3}
Y_2	{0.4, 0.3}	{0.8}	{0.4, 0.3, 0.2}
Y_3	{0.8}	{0.7, 0.6, 0.5}	{0.2, 0.1}

We utilize the WHFHWA operator (Eq. (21)) (suppose that $\theta = 1$) to aggregate all the preference values $\tilde{r}'_{ij}$ ($j = 1,2,3$) in the *i*th line of R' and then derive the overall performance value $\tilde{r}'_i$ ($i = 1,2,3$) of the alternative Y_i ($i = 1,2,3$):

$$\tilde{r}'_1 = \begin{Bmatrix} 0.5528, 0.4708, 0.5150, 0.4262, 0.5218, 0.4342, \\ 0.4814, 0.3864, 0.4949, 0.4024, 0.4523, 0.3519 \end{Bmatrix}$$

$$\tilde{r}'_2 = \{0.5184, 0.4798, 0.4438, 0.4956, 0.4551, 0.4175\}$$

$$\tilde{r}'_3 = \{0.5662, 0.5399, 0.5405, 0.5126, 0.5196, 0.4904\}$$

According to Definition 3.2, we calculate the score values $s(\tilde{r}'_i)$ ($i = 1,2,3$) of $\tilde{r}'_i$ ($i = 1,2,3$) as:

$$s(\tilde{r}'_1) = 0.5282, \quad s(\tilde{r}'_2) = 0.4575, \quad s(\tilde{r}'_3) = 0.4684.$$

Since $s(\tilde{r}_1') > s(\tilde{r}_3') > s(\tilde{r}_2')$, then we get the ranking of the alternatives Y_i $(i = 1,2,3)$ as $Y_1 > Y_3 > Y_2$. Thus, the best alternative is Y_1.

From the results of calculations, one can find a difference in the ranking results derived in Examples 5.1 and 5.2. The reason is that Example 5.1 uses the WHFEs to represent the preference values, which consider the importance of all of the possible values for an alternative under an attribute; while, Example 5.2 uses the HFEs to represent the preference values, which do not consider the importance of all of the possible values for an alternative under an attribute.

In fact, suppose that ten decision makers are required to anonymously provide their evaluations about the alternatives with regard to the attributes. For the evaluations about the alternative Y_1 with regard to the attribute G_1, assume that three decision makers provide 0.6, three decision makers provide 0.5, the remaining four decision makers provide 0.4, and these ten decision makers cannot persuade each other to change their opinions. When we consider a MAGDM problem, if two or more decision makers who are familiar with this area give the same preferences, then their preferences will be close to group preference. In such cases, the value repeated many times may be more important than the one repeated only one time. Therefore, the most likely evaluation about the alternative Y_1 with regard to the attribute G_1 should be 0.4. For the evaluations about the alternative Y_1 with regard to the attribute G_2, assume that eight decision makers provide 0.6, the remaining two decision makers provide 0.4, and these ten decision makers cannot persuade each other to change their opinions. Then, the most likely evaluation about the alternative Y_1 with regard to the attribute G_2 should be 0.6. For the evaluations about the alternative Y_1 with regard to the attribute G_3, assume that three decision makers provide 0.5, the remaining seven decision makers provide 0.3, and these ten decision makers cannot persuade each other to change their opinions. Then, the most likely evaluation about the alternative Y_1 with regard to the attribute G_3 should be 0.3. If we use the WHFHWA operator (Eq. (21)) (suppose that $\theta = 1$) to aggregate 0.4, 0.6, and 0.3, then the most likely overall performance value of the alternative Y_1 should be 0.4024:

We can easily see that the score value $s(\tilde{r}_1) = 0.4497$ of the alternative Y_1 obtained in Example 5.1 is much closer to the most likely overall performance value 0.4024 than the score value $s(\tilde{r}_1') = 0.5282$ of the alternative Y_1 obtained in Example 5.2. As a result, the WHFEs are more reasonable and reliable than the HFEs in some practical applications.

6 Conclusions

Considering that the classical hesitant fuzzy set does not consider the importance of several possible membership degrees of each element, in this paper, we have proposed a new generalization of the classical hesitant fuzzy set, which we call the WHFS. The WHFS adds a weight vector to several possible membership degrees of each element of the classical hesitant fuzzy set, which denotes the importance of several possible membership degrees. Then, based on Archimedean t-conorm and t-norm, we have defined some operational laws for WHFEs and studied their properties, based on which, we have developed two weighted hesitant fuzzy aggregation operators, including the ATS-WHFWA and ATS-WHFWG operators, and investigated some desired properties of two new operators. Furthermore, when the additive generator g is assigned different forms, some special cases of two new operators have been obtained, such as the WHFWA, WHFEWA, WHFHWA, WHFFWA, WHFWG, WHFEWG, WHFHWG, and WHFFWG operators. Finally, we have developed an approach based on the proposed operators for multi-criteria decision making with weighted hesitant fuzzy information, and the proposed operators and approach have been illustrated by a practical example involving a detailed analysis of the variation trend of the score functions and rankings of the alternatives with respect to the parameter θ.

Acknowledgements

The authors thank the anonymous referees for their valuable suggestions in improving this paper. This work is supported by the National Natural Science Foundation of China (Grant Nos. 71271070 and 61375075) and the Natural Science Foundation of Hebei Province of China (Grant Nos. F2012201020 and A2012201033).

Competing Interests

Authors have declared that no competing interests exist.

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