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General Equilibria

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Abstract

This research work is devoted to general equilibria in abstract spaces, through the agency of the ideal points, fixed points of multifunctions, Isac's cones, Choquet boundaries and applied mathematics for academic and industrialists engaged in eco - efficiency. It can be also considered as a short survey of the recent results obtained in this field.

Keywords: Ideal point, multifunction, Isac's cone, Choquet boundaries.

1 Introduction

There exists a great variety of definitions for the equilibria in: Chemistry, Economics, Physics, Life Sciences and Allied Applications like Physiology, Medicine, Politics and so on. But what it means in Mathematics? In this research work we present properties of the equilibria (ideal or critical) points for the most important class of generalized dynamical systems through the agency of the connections with the Efficiency and Choquet boundaries.Thus, Section 2 is devoted to the study of the existence and the properties for the equilibrium points sets. In Section 3 two important coincidence results between the (approximate) equilibrium points sets and the Choquet boundaries are given. Section 4 comprises a new approximate modality for the equilibrium points sets and the last section includes two applications. In order to simplify the basic references we refer the reader only to [1–3].

2 Existence of Equilibria Points under Completeness

In this section we study the existence of critical points for a class of generalized dynamical systems in separated locally convex spaces ordered by (weak) supernormal cones, introduced by G. Isac in 1983 using the weak completeness. We note that the weakly complete cones are very important in functional analysis and potential theory.

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Let X be a real Hausdorff locally convex space with the topology induced by a family $P = \{P_a : a \in I\}$ of seminorms, ordered by a convex cone K and its topological dual space X^* . X^* .

Definition 2.1 *K is called Isac's (nuclear or supernormal) cone if for every* $p \in P$ *their exists* * such that $p(x) \le f(x)$ for all $x \in$ *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
P = { P_a : $\alpha \in I$ } of seminorms, ordered by a convex cone K and its topological dual space X^{*}.
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zally convex space with the topology induced by a family

dered by a convex cone K and its topological dual space X^{*}.

(*nuclear or supernormal*) *co*

Many examples of such cones as these, their importance for Pareto efficiency and full nuclearity are exhibited in and so on.

Let A be a non-empty set in a Hausdorff locally convex space X ordered by a convex cone K .

Definition 2.2. *A set valued map* $\Gamma: A \to 2^A$ *is called a generalized dynamical system if* $\Gamma(a)$ *is non-empty for every* $a \in A$.

Let $\Gamma: A \to 2^A$ be the generalized dynamical system defined by $\Gamma(a) = A \cap (a - K), a \in A$.

Definition 2.3. We say that a_0 is an equilibrium (ideal or critical point) for Γ or a minimal *efficient (Pareto minimum) point for A with respect to K*, *in notation*, $a_0 \in MIN_{\kappa}(A)$ *if it satisfies one of the following equivalent conditions; British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 Aausdorff locally convex space with the topology induced by a family

reminorms, ordered by a convex cone K and its topological dual space X². **Example 3** *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014***

X be a real Hausdorff locally convex space with the topology induced by a family

{** P_a **:** $\alpha \in I$ **} of seminorms, ordered by a convex c** *^A ^K a MIN A* ⁰ *^K Reneralized dynamical system if* $I(a)$ *K* and by $\Gamma(a) = A \cap (a - K), a \in A$.
 A or critical point) for Γ *or a minimal* to K , *in notation*, $a_0 \in MIN_K(A)$ *if it*
 K Γ *is in notation* $A_0 \in MIN_K(A)$ *K K* coints give

- *(i)*
- *(ii)*
- (*iii*) $(A+K)\bigcap (a_0-K)\subseteq a_0+K$,
- (iv) $K \cap (a_0 A K) \subset -K$.

The *maximal efficient points* are defined similarly by replacing the convex cone K with $-K$. For this reason, in the sequel we consider the equilibrium points given by the efficient points as minimal. The term of efficient point will be frequently used instead of the equilibrium point in order to support the strong connection with the efficiency. *A* a ϵ *A* of seminorms, ordered by a convex cone *K* and its topological dual space *X*^{*}.
 A A *A K is called lsac's (nuclear or supernormal) cone <i>if for* every $p \in P$ *their exists* α *k at a t A a* $A \rightarrow 2^A$ be the generalized dynamical system defined by $\Gamma(a) = A \cap (a - K), a \in A$.
 an 2.3. We say that a_0 is an equilibrium (ideal or critical point) for Γ or a minimal (Pareto minimum) point for A with respect

We recall that every supernormal cone is pointed, that is, $K \cap (-K) = \{a_0\}$ and, in any such a case as this, $a_0 \in MIN_{\kappa}(A)$ if and only if $A \cap (a_0 - K) = \{a_0\}$, or equivalently $K \cap (a_0 - K) = \{0\}$.

The problem consists to apply the mathematical modelling and the corresponding numerical simulation in this way to the real processes.

Theorem 2.1. *If* $(X, P = \{p_a : a \in I\})$ *is a separated locally convex space with the topology generated by a family P of seminorms and A is a non-empty complete set in X such that for every* $p_a \in P$ *there exists a lower semicontinuous function* $\varphi_a : A \to R_+$ *with* $p_{\alpha}(x-y) \leq \varphi_{\alpha}(x) - \varphi_{\alpha}(y)$, $\forall x \in A, y \in \Gamma(x)$, then $MIN_{K} \neq \varphi$. *Inimum) point for A* with respect to *K*, in notation, $a_0 \in MIN_{\kappa}(A)$ *if it*
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 $A \subseteq -K$,
 $\sum (a_0 - K) \subseteq a_0 + K$,
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 X $A \cap K = -K$,
 X and *p* $P_1 = \sum_{n=1}^{n} P_2$ $P_3 = \sum_{n=1}^{n} P_4$ $P_5 = a_6 + K$,
 iii $\int (A_4 - A) \subseteq -K$.

The maximal efficient points are defined similarly by replacing the convex cone K with $-K$.

The maximal efficient points are defined simil The importance of Isac's cones for the existence and the domination property of the equilibrium points is also illustrated by the following results.

Theorem 2.2. Let $A \subseteq B \subseteq A + K$. If K is supernormal and $B \cap (A_0 - K)$ is bounded and *complete for some non-empty set* $A_0 \subseteq A$, then $MIN_K(A) \neq \emptyset$. *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 A ac's cones for the existence and the domination property of the equilibrium

ted by the following results.
 A \subseteq *B* \subseteq *A* + *K*. *If A A* ⁰ . *MIN A ^K*

Corollary 2.2.1. Let $A \subseteq B \subseteq A + K$. If K is weakly supernormal and $B \cap (A_0 - K)$ is bounded *and weakly complete for some non-empty set* $A_0 \subseteq A$, then $MIN_{k}(A) \neq \emptyset$.

In particular, if K is a weakly supernormal cone in *X* and $A \cap (a - K)$ or $(A + K) \cap (a - K)$ is *bounded and weakly complete for some* $a \in A$ *, then* $MIN_{r}(A) \neq \emptyset$ *. When the boundedness and weak completeness properties hold for every* $a \in A$, then we have the following domination *property* $A \subseteq MIN_K(A) + K$. *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
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A B \subseteq *B* \subseteq *A* + *K . If K is supernormal A A* ⁰ . *MIN A ^K British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
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<i>then MIN_K* (*A*) $\neq \phi$.
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Let $A \subseteq B \subseteq A + K$. If K is weakly supernormal and $B \cap (A_0 - K)$ is bound **A.** Let $A \subseteq B \subseteq A + K$. If K is supernormal and $B \cap (A_0 - K)$ is bounded and

some non-empty set $A_0 \subseteq A$, then $MN_k(A) \neq \emptyset$.
 A.2.1. Let $A \subseteq B \subseteq A + K$. If K is weakly supernormal and $B \cap (A_0 - K)$ is bounded

complete for s *Let* $A \subseteq B \subseteq A + K$. If K is weakly supernormal and $B \cap (A_0 - K)$ is bounded
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K is a weakly supernormal cone in X and $A \cap (a - K)$ or $(A + K) \cap (a - K)$ is
akky complet *K*, if *K* is a weakly supernormal cone in *X* and $A \cap (a-K)$ or $(A+K) \cap (a-K)$ is
d weakly complete for some $a \in A$, then $MIN_{K}(A) \neq \phi$. When the boundedness and
deteness properties hold for every $a \in A$, then we have the *P*, *A K* is a weakly supernormal cone in *X* and $A \cap (a - K)$ or $(A + K) \cap (a - K)$ *is* d weakly complete for some $a \in A$, then $MN_k(A) \neq \emptyset$. When the boundedness and deteness properties hold for every $a \in A$, then we have

Since in every separated locally convex space any normal cone is weakly supernormal, the conclusion of the above corollary remains valid whenever K is normal. Using this remark one obtains the next existence results for efficient points.

Corollary 2.2.2. $MIN_{K}(A) \neq \emptyset$ *if one satisfies one of the following conditions;*

- iii) (i) *K* is closed, normal, weakly complete and *A* is weakly closed such that $A \cap (a - K)$ *is bounded for some* $a \in A$. We also have $A \subseteq MIN_{\kappa}(A) - K$ *if, under the above hypotheses,* $A \cap (a - K)$ *in bounded for every* $a \in A$, *a* $A \subseteq MIN_k(A) + K$.
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the next existence results for efficient points.
 c 2.2.2. $MIN_k(A) \neq \emptyset$ fone satisfies o *M_N* (*A*) $\neq \emptyset$ *fore satisfies one of the following conditions;

<i>Closed, normal, weakly complete and A is weakly closed such that*
 K) *K* bounded for some $a \in A$. *We also have* $A \subseteq MIN_{\kappa}(A) - K$ *if,*
 A is bou K is closed, normal, weakly complete and *A* is weakly closed such that \mathbb{R}^d $\{N(a - K)$ *is bounded for some* $a \in A$ *. We also have* $A \subseteq \text{MIN}(A) - K$ *if,* \mathbb{R}^d *,* $\det P$ *<i>he above hypotheses,* $A \cap \{a - K\}$ *in bounded*
- iiii) (ii) *is closed, normal and is bounded and weakly complete. The domination property holds again,*
- iiiii) K is closed, normal, weakly complete and $A + K$ is weakly closed such that $(A+K)\cap (a-K)$ *is bounded for some* $a \in A$ *.*

The domination property $A \subseteq MIN_K(A) + K$ *holds if, in addition* $(A+K) \cap (a-K)$ *is bounded for any* $a \in A$.

Remark. 2.1. It is clear that all the above results remain valid if one replaces the hypothesis of normality on K with the weak normality.

Corollary 2.2.3. If A is a non-empty, bounded and closed subset of X and K is well based *(that is, generated by a non-empty convex, bounded set which does not contain the origin in its closure) by a complete set, then* $MIN_{K}(A) \neq \emptyset$ and $A \subseteq MIN_{K}(A) + K$.

Corollary 2.2.4. *If is a non-empty, bounded and closed subset of a Banach space ordered by a convex cone K* well based by a closed set, then $MIN_{K}(A) \neq \emptyset$ and $A \subseteq MIN_{K}(A) + K$.

Remark 2.2. Since in a normed space a convex cone is supernormal if and only if it is well based, the last corollary offers some conclusion whenever K is supernormal. Moreover, the existence results given in this section show also the possibility to use the weakly complete cones for the study of the equilibrium (Pareto type optimization) in separated locally convex spaces. For comprehensive bibliography, we refer [6] and references therein. *Eritish Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 Mark 2.2. Since in a normed space a convex cone is supernormal if and only if it is well based,

Last conollary offers some conclusion whenever

3 Coincidence Results between Equilibria Points Sets, Choquet Boundaries and Related Topics

In our all futher considerations we suppose that (E, τ) is a Hausdorff locally convex space, where τ denotes its topology, K is a closed, convex pointed cone in E and ε is an arbitrary element of $K \setminus \{0\}$. On the vector space E we consider the usual order relation \leq_K associated with K as follows: for $x, y \in E$ one defines $x \leq_{K} y$ iff $y \in x + K$.

Clearly, this order relation on E is closed, that is, the set G_k given by $G_k = \{(x, y) \in E \times E : x \leq_k y\}$ is a closed subset of $E \times E$. Also the set $G_{\varepsilon+K} = \{(x, y) \in E \times E : y \in x + \varepsilon + K\}$ is closed in $E \times E$ endowed with the usual product topology. **British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014

Remark 2.2, Since in a normed space a convex cance is supernormal if and only if it is well based,

he last coollary offers some conclusion wheneve Example 10 A** *K* is the special of $E \times E$ **A** *K K* **Chooper** *A**K* **Chooper** *A* **A Chooper** *A* **Chooper** Altertions we suppose that (E, τ) is a Hausdorff locally convex space, where the interconsiderations we **3 Coincidence Results between Equilibria Points Sets, Choquet

Boundaries and Related Topics**

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Clearly, this order relation on *F* is closed, that is, the set G_E given by $G_{\pi}E = \{(x, y) \in E \times E : x \leq_E y\}$ is closed subset of $E \times E$. Also the set $G_{\pi, K} = \{(x, y$ *K* is, the set G_K given by

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 $a_0 \in A$ will be called an ε -critical point
 $a_0 \in A$ will be called an ε -critical point
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Definition 3.1. *If A* is a non-empty subset of *E*, then $a_0 \in A$ will be called an ε -critical point *(or, - minimal element, - efficient point, Pareto - efficient point, - near to minimum point) of A* with respect to *K* if there exists no $a_0 \in A$ such that

 $a_0 - a - \varepsilon \in K$, that is, $a_0 - a - \varepsilon \in K \cap A = \phi$.

The ε -critical points set of A with respect to K will be denoted by ε -MIN_K(A) $(\varepsilon - \textit{eff}(A,K)).$

Remark 3.1. It is clear that the concept of the ε -efficient points does not include the notion of efficient point, will be denoted by ε - MIN_K (A)

efficient points does not include the notion of
 $K_K(A) = \bigcap_{\varepsilon \in K \setminus \{0\}} [\varepsilon - MIN_K(A)].$

generalization of the approximative Pareto

non-empty subset of $K \setminus \{0\}$. In this way it is

$$
MIN_{K}(A) \subseteq \varepsilon - MIN_{K}(A), \forall \varepsilon \in K \setminus \{0\} \text{ and } MIN_{K}(A) = \bigcap_{\varepsilon \in K \setminus \{0\}} \big[\varepsilon - MIN_{K}(A)\big].
$$

Remark 3.2. A very interesting and important generalization of the approximative Pareto efficiency can be obtained by replacing ε with a non-empty subset of $K \setminus \{0\}$. In this way it is shown that the existence of this new type of efficient points for lower bounded sets characterizes the semi-Archimedian ordered vector spaces and the regular ordered locally convex spaces. oted by ε - $MIN_K(A)$

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subset of $K \setminus \{0\}$. In this way it is *Et of E*, then $a_0 \,\epsilon A$ will be called an ε -critical point
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concept of the ε - efficient points does not include the notion of
 $\varepsilon K \setminus \{0\}$ and $MIN_K(A) = \bigcap_{\varepsilon$ *xoint)* of *A* with respect to *K* if there exists no $a_0 \,\epsilon A$ such that
 $a_0 - a - \varepsilon \in K$, that is, $a_0 - a - \varepsilon \in K \cap A = \phi$.

The *ε* - critical points set of *A* with respect to *K* will be denoted by *ε* - *MIN_K*(*A*

Definition 3.2 *A real function* $f: E \to \square$ *is caled* $\varepsilon + K$ - *increasing if* $f(x_1) \ge f(x_2)$ *whenever* $x_1, x_2 \in E$ and $x_1 \in x_2 + \varepsilon + K$.

For a non-empty and compact subset X of E we recall some basic consideration in potential theory concerning the Choquet boundary of X with respect to a convex cone of continuous functions on X. Thus, we remember that if S is a convex cone of real continuous functions on X such that the constant function on X belong to S, it is min-stable (i.e., for every $f_1, f_2 \in S$ it follows inf $(f_1, f_2) \in S$) and it separates the points of X, then on the set $M_+(X)$ of all positive Radon measures on X we associate the following order relation; *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
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Choquet boundary of X with respect to a convex cone of continuous

we rem For a non-empty and compact subset *X* of *F* we recall some basic consideration in potential
theory concerning the Choquet boundary of *X* with respect to a convex cone of continuous
functions on *X*. Thus, we remember t

if $\mu, \nu \in M_+(X)$ then $\mu \leq s \nu$ iff $\mu(s) \leq \nu(s)$ for all $s \in S$. A measure $\mu \in M_+(X)$ is minimal with respect to the above order relation if for any continuous function $f: X \to \square$ we have $\mu(Q,f) = \mu(f)$, where $\mu(Q,f) = \inf \{ s \in S : f \leq s \}$. Particularly, if $x \in X$, then the Dirac measure ε_x is minimal iff $\varepsilon_x(Q_s f) = \varepsilon_x(f)$, that is, $Q_s f(x) = f(x)$ for every continuous function $f: X \to \square$. *A X ^S x A s x X A*\ 0.) and it separates the points of *X*, then on the set $M_+(X)$ of all positive
we associate the following order relation;
 $\mu \leq s y$ iff $\mu(s) \leq v(s)$ for all $s \in S$. A measure $\mu \in M_+(X)$ is minimal
ove order relation if fo

The set of all points $x \in X$ such that ε_x is minimal measure with respect to \leq_s is named the Choquet boundary of X with respect to S and it is denoted by $\delta_s X$. Hence, if $C(X)$ is the usual Banach space of all real continuous functions on X , then

A closed set $A \subseteq X$ is called S - absorbent if $x \in A$ and $\mu \leq_{s} \varepsilon_{x}$ implies $\mu(X \setminus A) = 0$. The trace on $\delta_s X$ of the topology on X in which the closed set coincides with X or with the absorbent subset of X contained in $\{x \in X : \exists s \in S \text{ with } s(x) < 0\}$ is named usually the Choquet topology on δX . $\delta_{\scriptscriptstyle s} X$.

An important connection between vector optimization and potential theory is the next coincidence of efficient points sets and Choquet boundaries in separated locally convex spaces which cannot be obtained as a consequence of the axiomatic potential theory.

Theorem 3.1. $MIN_{K}(X)$ coincides with the Choquet boundary of X with respect to the convex *cone of all real continuous functions which are increasing with respect to the order relation* \leq_{κ} . *Consequently, the set* $MIN_{K}(X)$ endowed with the trace topology τ_{X} induced on X by τ is a *Baire space. Moreover, if X* is metrizable, then $MIN_K(X)$ is a G_s -set in (X, τ_X) . \rightarrow \sqcup .
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ary of *X* with respect to *S* and it is denoted by $\delta_x X$. Hence, if $C(X)$ is the

ace of *K* **Example 12 K** *K C(X) X*th respect to *S* and it is denoted by $\delta_i X$. Hence, if $C(X)$ is the sal continuous functions on *X*, then
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Iled *S* - absorbent if $x \in A$ and $\mu \leq_x \delta_x$ implies $\mu(X \setminus A) = 0$. The
logy on *X* in which **Example 3** $A \subseteq X$ is called *S* - absorbent if $x \in A$ and $\mu \leq x$, implies $\mu(X \setminus A) = 0$. The set $A \subseteq X$ is called *S* - absorbent if $x \in A$ and $\mu \leq x$, capitolis with *X* or with the slosed set coincides with *X* or subset of *X* contained in $\{x \in X : \exists s \in S \text{ with } s(x) < o\}$ is named usually the pology on $\delta_x X$.
tant connection between vector optimization and potential theory is the next coincidence the top to the external of the axiom

Consequently,

Corollary 3.1.1

- iii) $MIN_{\kappa}(X)$ and $MIN_{\kappa}(X) \cap \{x \in X : s(x) \le 0\}$ ($s \in S$) are compact sets with *respect to Choquet's topology;*
- iiii) $MIN_{\kappa}(X)$ is a compact subset of X.

Remark 3.3. In the conditions of Theorem 3.1 let us consider, on $MIN_{\kappa}(X)$ endowed with the trace topology also denoted by τ_X , the following game between parteners A and B: each partener succesively chooses a non-empty set belonging to τ_X such that the player A makes the first choice and each player must choose a set in τ_X which should be included into the previous chosen set of the other player. *er Science 4(8), 1048-1073, 2014*
MIN_K (*X*) endowed with the parteners *A* and *B*: each that the player *A* makes the be included into the previous **Example 10** British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014
 ark 3.3. In the conditions of Theorem 3.1 let us consider, on $MN_k(X)$ endowed with the

topology also denoted by τ_x , the following *British Journal of Mathematics* & *Computer Science* $4(8)$, $1048-1073$, 2014
 3. In the conditions of Theorem 3.1 let us consider, on $MIN_x(X)$ endowed with the

negy also denoted by τ_x , the following game between

Let $G_1, G_1, G_2, G_2, \ldots, G_n, G_n, \ldots$ be the succesive options of the two players, $(G_1, G_2, \ldots, G_n, \ldots)$ represent the option expressed by A and $(G_1, G_2, ..., G_n, ...)$ the option made by B. One says that the player B wins if no matter the way A plays, he is able to make an option so that $\bigcap_{x\in\Box^*} G_n \neq \phi.$

Theorem 3.1 together with Choquet's results concerning the properties of the Choquet boundary shows that the above game on $MIN_K(X)$ is won by the player B.

Remark 3.4. Under the hypotheses of Theorem 3.1, the set $\mathit{eff}(X,K)$ coincides with the Choquet boundary of X only with respect to the convex cone of all real, continuous and K - increasing functions on X. Thus, for example, if X is a non-empty, compact and convex subset of E then, the Choquet boundary of X with respect to the convex cone of all real, continuous and concave functions on X coincides with the set of all extreme points for A , that is, with the set of elements $x \in X$ such that if $x_1, x_2 \in X$, $\lambda \in (0,1)$ and $x = \lambda x_1 + (1-\lambda)x_2$, then $x = x_1 = x_2$. But, it is easy that, even in infinite dimensional cases, an extreme points for a compact convex set is not necessary an efficient points and conversely. *Fritish Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

of Theorem 3.1 let us consider, on $MN_k(X)$ endowed with the
 $y \tau_x$, the following game between parteners A and B: each

non-empty set belonging to *xiac* topology also denoted by r_x , the following game between parteners A and B : each
 xiacret exceesively chooses a non-empty set belonging to r_x such that the player A makes the
 xis x G_1 , G_2 , $G_$ *N* expressed by *A* and $(G_1, G_2, ..., G_n, ...)$ the option made by *B*. One says that if no matter the way *A* plays, he is able to make an option so that
the with Choquet's results concerning the properties of the Choquet bound requenter with Choquet's results concerning the properties of the Choquet boundary
above game on $MIN_K(X)$ is won by the player *B*.

Under the hypotheses of Theorem 3.1, the set $\epsilon df(X, K)$ coincides with the Choquet
 X onl *X*: If the master with Choquet's results concerning the properties of the Choquet boundary is that the above game on $MN_k(X)$ is won by the player *B*.
 Ark 3.4. Under the hypotheses of Theorem 3.1, the set $\epsilon f f(X, K)$ co

The following result extends Theorem 3.1 for ε - effuciency.

Theorem 3.2. *If X* is a non-empty subset of *E*, then the set ε -eff (X, K) coincides with the *Choquet boundary of X with respect to the convex cone all* $\varepsilon + K$ *- increasing real continuous functions on X*. Consequently, the set ε – eff (X, K) endowed with the trace topology is a Baire *space and if* (X, τ_X) *is metrizable, then* ε - *eff* (X, K) *is a* G_{δ} - *subset of* X.

4 A Generalized Modality for the Equilibria Points Sets

Let X be a vector space ordered by a convex cone K, K_1 a non-void subset of K and A a nonempty subset of X. The following definition introduces a new concept of (approximate) Pareto type efficient points which, particularly, leads to the well known notion of Pareto efficiency (in fact, the generalization in abstract spaces of the finite dimensional notion as we shall see in the next considerations.

Definition 4.1. We say that $a_0 \in A$ is a K_1 – equilibrium(K_1 – Pareto minimal or efficient) point *of A*, in notation, $a_0 \in \text{eff}(A, K, K_1)$ (or $a_0 \in MIN_{K+K_1}(A)$) if it satisfies one of the following *equivalent conditions:* **British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014**
 inition 4.1. We say that $a_0 \in A$ is a K_1 -equilibrium(K_1 - Pareto minimal or efficient) point
 A, in notation, $a_0 \in \text{eff}(A, K, K_1)$ (or *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 n 4.1. We say that $a_0 \,\in A$ is a K_1 - equilibrium(K_1 - Pareto minimal or efficient) point

notation, $a_0 \in \text{eff}(A, K, K_1)$ (or $a_0 \in MIN_{K+$

$$
(i) \qquad A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1;
$$

$$
(ii) \qquad (K+K_1) \cap (a_0-A) \subseteq -K-K_1;
$$

In a similar manner one defines the Pareto (maximal) efficient points by replacing $K + K₁$ with

Remark 4.1. $a_0 \in \text{eff}(A, K, K_1)$ iff it is a fixed point for the multifunction $F: A \rightarrow 2^A$ defined by $F(t) = \{a \in A : A \cap (a - K - K_1) \subseteq t + K + K_1\}$, that is, an *equilibrium point* for the generated generalized dynamical system **Example 10** *Buttish Journal of Mathematics & Computer Science 4(8), 1045-1073, 2014*
 Perfinition 4.1. *We say that* $a_0 \in A$ *is a* K_1 - equilibrium(K_1 - Pareto minimal or efficient) point

equivalent condition *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 inition 4.1. *We say that* $a_0 \in A$ is aK_1 -equilibrium(K_1 - Pareto minimal or efficient) point

4, in notation, $a_0 \in \text{eff}(A, K, K_1)$ (or *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 Definition 4.1. We say that $a_6 \in A$ is a K_1 -equilibrium(K_1 -Pareto minimal or efficient) point

quivalent conditions:
 $f(A_1 \text{ in notation, } a_5 \in$ *K*_{*K*_{*K*}_{*K*</sup>_{*K*}^{*K*}_{*K*}^{*K*}*K*^{*K*}}}

 $\Gamma = F$. Consequently, for the existence of the Pareto type efficient points it can applied appropriate fixed point theorems concerning the multifunctions.

Remark 4.2. It is well known that whenever $K_1 \subset K \setminus \{0\}$, the existence of this new type of efficient points for lower bounded sets characterizes the semi Archimedian ordered vector spaces and the regular ordered locally convex spaces.

Remark 4.3. When K is pointed, that is, $K \cap (-K) = \{0\}$, then $a_0 \in \text{eff}(A, K, K)$ means that $A \cap (a_0 - K - K_1) = \emptyset$ or, equivalently, $(K + K_1) \cap (a_0 - A) = \emptyset$ for $0 \notin K_1$ and $A \cap (a_0 - K - K_1) = \{a_0\}$, respectively, if $0 \in K_1$. Whenever K is pointed and $K_1 = \{0\}$, from Definition 4.1, one obtains the well known usual notion of Pareto (minimal, efficient, optimal or admissible) point, abbreviated $a_0 \in \text{eff}(A, K)$ (or $a_0 \in \text{MIN}_K(A)$), that is, satisfying the next equivalent properties: *K*, $B = K_1 \leq a_0 + K + K_1$;
 $a_0 - A \leq -K - K_1$;

the defines the Pareto (maximal) efficient points by replacing $K + K_1$ with
 $K(A, K, K_1)$ iff it is a fixed point for the multifunction $F : A \rightarrow 2^d$ defined
 $\bigcap (a - K - K_1) \subseteq t + K +$ *A* $\bigcap (a_n - K - K_1 \bigsubseteq a_n + K + K_1;$
 A $(K + K_1) \bigcap (a_n - A \bigsubseteq -K - K_1;$
 A a similar manner one defines the Pareto (maximal) efficient points by replacing $K + K_1$ with
 $-(K + K_1).$
 A Remark 4.1. $a_n \in eff(A, K, K_1)$ iff it is a fixe *A* $(K + K_1) \cap (a_0 - A) \subseteq -K - K_1;$
 A a similar manner one defines the Pareto (maximal) efficient points by replacing $K + K_1$ with
 $-(K + K_1).$
 Remark 4.1. $a_n \in eff(A, K, K_1)$ iff it is a fixed point for the multifunction $F : A$ *a* iff it is a fixed point for the multifunction $F : A \rightarrow 2^A$ defined K_1) $\subseteq t + K + K_1$, that is, an *equilibrium point* for the generated existence of the Pareto type efficient points it can applied concerning the multi *A a K a* 0 0 ; *A* $\bigcap (a - K - K_1) \subseteq t + K + K_1\}$, that is, an *equilibrium point* for the generated
ical system
mtly, for the existence of the Pareto type efficient points it can applied
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 K, for the existence of the Pareto type efficient points it can applied

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well known that whenever $K_1 \subset K \setminus \{0\}$, the existence of this new type of

lower bou finity, for the existence of the Pareto type efficient points it can applied
oint theorems concerning the multifunctions.
well known that whenever $K_1 \subset K \setminus \{0\}$, the existence of this new type of
lower bounded sets c point, abstracted $u_0 \in \mathcal{E}$ $H(X, K)$ (or $u_0 \in \text{MIN}_K(X)$), that is, satisfying the

reperties:
 $A \cap (a_0 - K) = \{a_0\};$
 $A \cap (a_0 - K \setminus \{0\}) = \emptyset;$
 $K \cap (a_0 - A) = \{0\};$
 $(K \setminus \{0\}) \cap (a_0 - A) = \emptyset$

ce that
 $\bigcap_{\{0\} \in K_2 \subseteq K}$ efficient points for lower bounded sets characterizes the semi-Archimedian ordered vector spaces

ond the regular ordered locally convex spaces.
 Remark 4.3. When $\alpha_a \in \text{eff}(A, K, K)$ means that
 $A \cap (a_a - K - K,) = \emptyset$ or, eq It is well known that whenever $K_1 \subset K \setminus \{0\}$, the existence of this new type of
for lower bounded sets characterizes the semi Archimedian ordered vector spaces
ordered locally convex spaces.
When K is pointed, that is,

- (i)
- (ii) $A \cap (a_0 K \setminus \{0\}) = \emptyset;$
- (iii) $K \bigcap (a_0 A) = \{0\};$
- (iv) $(K\setminus\{0\})\bigcap (a_{0}-A)=\emptyset$

and we notice that

$$
eff(A, K, K_1) = eff(A, K) = \bigcap_{\{0\} \neq K_2 \subseteq K} eff(A, K, K_2).
$$

It is clear that for any $\varepsilon \in K \setminus \{0\}$, taking $K_1 = \{\varepsilon\}$, it follows that $a_0 \in \text{eff}(A, K, K_1)$ if and only if $A \cap (a_0 - \varepsilon - K) = \emptyset$. In all these cases, the set $\text{eff}(A, K, K)$ was denoted by and it is obvious that $\operatorname{eff}(A, K) =$ *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 $\varepsilon \in K \setminus \{0\}$, taking $K_1 = \{\varepsilon\}$, it follows that $a_0 \in \text{eff}(A, K, K_1)$ if and
 $f(x) = \emptyset$. In all these cases, the set $\text{eff}(A, K, K_1)$ was de *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

that for any $\varepsilon \in K \setminus \{0\}$, taking $K_1 = \{\varepsilon\}$, it follows that $a_0 \in \text{eff}(A, K, K_1)$ if and
 $A \cap (a_0 - \varepsilon - K) = \emptyset$. In all these cases, the s *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 t is clear that for any $\varepsilon \in K \setminus \{0\}$, taking $K_i = \{\varepsilon\}$, it follows that $a_0 \in \mathcal{C}f(A, K, K_1)$ if and

only if $A \cap (a_0 - \varepsilon - K) = \emptyset$. In Science 4(8), 1048-1073, 2014
 $a_0 \in \text{eff}(A, K, K_1)$ if and
 $,K, K_1$ was denoted by
 $\bigcup_{\{0\}} [\varepsilon - \text{eff}(A, K)].$

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it follows that $a_0 \in \text{eff} (A, K, K_1)$ if and

the set $\text{eff} (A, K, K_1)$ was denoted by
 $\text{eff} (A, K) = \bigcap_{\varepsilon \in K \setminus \{0\}} [\varepsilon - \text{eff} (A, K)].$

important connection between the stron ε puter Science 4(8), 1048-1073, 2014

that $a_0 \in eff(A, K, K_1)$ if and

ff (A, K, K_1) was denoted by
 $= \bigcap_{\varepsilon \in K \setminus \{0\}} [\varepsilon - eff(A, K)].$

connection between the strong

nment of ordered vector spaces, *Sih Journal of Mathematics* & *Computer Science* 4(8), 1048-1073, 2014
 S, taking $K_1 = \{e\}$, it follows that $a_0 \in eff(A, K, K_1)$ if and

n all these cases, the set $\text{eff}(A, K, K_1)$ was denoted by

and it is obvious that *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 Secar that for any $\alpha \in K \setminus \{0\}$, taking $K_1 = \{a\}$, it follows that $a_0 \in \theta f'(A, K, K_1)$ if and

if $A \cap (a_0 - \alpha - K) = \emptyset$. In all these cases, *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

at for any $\varepsilon \in K \setminus \{0\}$, taking $K_1 = \{z\}$, it follows that $a_0 \in eff(A, K, K_1)$ if and
 $|(a_0 - \varepsilon - K) = \emptyset|$. In all these cases, the set $eff(A, K, K$ *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 a $\pi x \in K \setminus \{0\}$, taking $K_1 = \{\varepsilon\}$, it follows that $a_0 \in \varepsilon f(f(A, K, K_1))$ if and $-\varepsilon - K$) = \emptyset . In all these cases, the set $\varepsilon f(f(A$ *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

clear that for any $\varepsilon \in K \setminus \{0\}$, taking $K_1 = \{\varepsilon\}$, it follows that $a_0 \in \mathcal{C}f(A, K, K_1)$ if and

if $A \cap (a_0 - \varepsilon - K) = \emptyset$. In all these c ar that for any $\varepsilon \in K \setminus \{0\}$, taking $K_1 = \{e\}$, it follows that $a_0 \in \mathcal{C}[A, K, K_1)$ if and $A \cap (a_0 - \varepsilon - K) = \emptyset$. In all these cases, the set $\mathcal{C}[A, K, K_1]$ was denoted by A, K) $(or \varepsilon - M/N_K(A)$ and it is obvious that for any $s \in K \setminus \{0\}$, taking $K_i = \{e\}$, it follows that $a_n \in \mathcal{gf}(A, K, K_i)$ is added by K and $A \cap (a_n - e - K) = \emptyset$. In all these cases, the set $\mathcal{ef}(A, K, K_i)$ was denoted by K , $\bigcirc (or s - M dK) \in \mathcal{af}(A, K) = \bigcap_{n \in \{$

Remark 4.4. The following theorem offers the first important connection between the strong optimization and the (approximate) Pareto efficiency in the environment of ordered vector spaces, described initially on the previous Definition 4.1.

Theorem 4.1. *If we denote by* $S(A,K,K_1) = \{a_1 \in A : A \subseteq a_1 + K + K_1\}$ *and* $S(A,K,K_1) \neq \emptyset$, *then* $S(A,K,K_1) = eff(A,K,K_1)$.

Proof.

Clearly, $S(A,K,K_1) \subset eff(A,K,K_1)$.

Indeed, if $a_0 \in S(A, K, K_1)$ and $a \in A \cap (a_0 - K - K_1)$ are arbitrary elements, then $a \in a_0 + K + K_1$, that is, $a_0 \in \text{eff}(A, K, K_1)$, by virtue of (i) in Definition 2.1. Suppose now that $\overline{a} \in S(A, K, K_1) \neq \emptyset$ and there exists $a_0 \in \text{eff}(A, K, K_1) \setminus S(A, K, K_1)$. *S* following theorem offers the first important connection between the strong
 A the previous Definition 4.1.
 S at denote by $S(A, K, K_*) = \{a_i \in A : A \subseteq a_i + K + K_i\}$ and $S(A, K, K_*) \neq \emptyset$,
 $= \text{eff}(A, K, K_*)$.
 $= \text{eff}(A, K, K_*)$.

From $\overline{a} \in S(A, K, K_1)$ it follows that $a_0 \in \overline{a} + K + K_1$, that is, $\overline{a} \in a_0 - K - K_1$, from which, since $\overline{a} \in A$ and $a_0 \in \text{eff}(A, K, K_1)$ we conclude that $\overline{a} \in a_0 + K + K_1$.

Therefore, $A \subseteq \overline{a} + K + K_1 \subseteq a_0 + K + K_1$, in contradiction with $a_0 \notin S(A, K, K_1)$ as claimed.

Remark 4.5. If $S(A, K, K_1) \neq \emptyset$, then $K + K_1 = K$ hence $eff(A, K, K_1) = eff(A, K)$. Indeed, let $a \in S(A, K, K)$. Then, $a \in a + K + K$, which implies that $0 \in K + K$.

Therefore, $K \subseteq K_1 + K + K = K_1 + K \subseteq K$. The above theorem shows that, for any non-empty subset of an arbitrary vector space, the set of all strong minimal elements with respect to any convex cone through the agency of every non-noid subset of it coincides with the corresponding set of Pareto (minimal) efficient points whenever there exists at least a strong minimal element, the result remaining obviously valid for the strong maximal elements and the Pareto maximal efficient points, respectively. Using this result and our abstract construction given in for the splines in H-locally convex spaces we concluded that the only best simultaneous and vectorial approximation for each element in the direct sum of a (closed) linear subspace and its orthogonal with respect to a linear (continuous) operator between two H-locally convex spaces is its spline function. We also note that it is possible to have $S(A, K, K_1) = \emptyset$ and $\text{eff}(A, K, K_1) = A$. Thus, for example, if one considers $X = R^n (n \in N, n \ge 2)$ endowed with the separated H-locally convex topology generated by the semi-norms $p_i: X \to R_+$, $p_i(x) = |x_i|, \forall x = (x_i) \in X$, $i = \overline{1, n}$, $K = R_{\perp}^{n}$, $K_1 = \{(0, ..., 0)\}$ and for each real number c we define *and* $4A$, \cdot in Exterior guotent order to \cdot rank \cdot and \cdot and \cdot and \cdot and \cdot and \cdot *A* (*A,K,K,)* = $\{a, \in A : A \subseteq a_1 + K + K_1\}$ and $S(A, K, K_1) \neq 0$, in the priviolar initially on the previous Definiti 1. *If* we denote by $S(A,K,K_1) = \{a_i \in A : A \subseteq a_i + K + K_i\}$ and $S(A,K,K_1) \in \mathcal{D}$.
 $(K,K_1) \in \mathcal{G}f(A,K,K_1)$.

if $a_0 \in S(A,K,K_1)$ and $a \in A \cap \{a_0 - K - K_1\}$ are arbitrary elements, then
 $K + K_1$, that is, $a_0 \in \mathcal{G}(A,K,K_1)$ by virtu *R, k, k*, *j*).

and $a \in A \cap (a_0 - K - K_1)$ are arbitrary elements, then
 $\operatorname{eff}(A, K, K_1)$, by virtue of (i) in Definition 2.1. Suppose now

there exists $a_0 \in \operatorname{eff}(A, K, K_1) \setminus (A, K, K_1)$.

by such that is $\overline{a_0} \in \overline{A}$ *i* $\Gamma(\vert \mathbf{Q}_0 - K - K_i \vert)$ are arbitrary elements, then
 b y virtue of (i) in Definition 2.1. Suppose now
 $o \in \text{eff}(A, K, K_1) \setminus S(A, K, K_1)$.
 $\overline{\tau} + K + K_1$, that is, $\overline{\alpha} \in a_0 - K - K_1$, from which,

clude that $\overline{\alpha} \in a$ *i* that $a \in S(A, K, K) \neq \emptyset$ and there exists $a_0 \in \mathbf{ef}(A, K, K)$. *S* { A, K, K , then $\overline{a} \in S(A, K, K)$ if follows that $a_0 \in \mathbf{ef}(A, K, K)$ *c* and $x_0 \in A + K + K$, $x_0 \in A, K + K$, $x_0 \in A, K$. *K* is $\overline{a} \in a_0 \in A, K \in \mathbf{f}(A$

British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014\n
$$
A_c = \left\{ (x_i) \in X : \sum_{i=1}^n x_i = c \right\}, \text{ then it is clear that } S\left(A_c, K, K_1 \right) \text{ is empty and}
$$
\n*eff*
$$
\left(A_c, K, K_1 \right) = A_c.
$$
\nIn all our further considerations we suppose that *X* is a Hausdorff locally convex space having the topology induced by family $P = \{ p_\alpha : \alpha \in I \}$ of seminorms, ordered by a convex cone *K* and its topological dual space *X*^{*}. In this framework, the next theorem contains a significant criterion

In all our further considerations we suppose that X is a Hausdorff locally convex space having the topology induced by family $P = \{p_a : a \in I\}$ of seminorms, ordered by a convex cone K and its topological dual space X^* . In this framework, the next theorem contains a significant criterion for the existence of the approximative Pareto (minimal) efficient points, in particular, for the usual Pareto (minimal) efficient points, taking into account that the dual cone of K is defined by $x^* \in X^*$: $x^*(x) \ge 0$, $\forall x \in K$ and its attached polar cone is $K^0 = -K^*$. The version for the (approximative) Pareto (maximal) efficient points is straightforward. *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 $A_c = \left\{ (x_i) \in X : \sum_{i=1}^n x_i = c \right\}$, then it is clear that $S(A_c, K, K_1)$ is empty and
 $\operatorname{eff}(A_c, K, K_1) = A_c$.

In all our further considerations we s *P p I* : *^K* **Example 10 A** *K* $K = \left\{ (x_i) \in X : \sum_{i=1}^{n} x_i = c \right\},$ then it is clear that $S(A_c, K, K_1)$ is empty and $\text{Aff}(A_c, K, K_1) = A_c$.
 $\text{and } \text{non-trative considerations we suppose that } X \text{ is a Hausdorff locally convex$ *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 $(\mathbf{X}, \mathbf{X}) = \mathbf{X}$, $\sum_{i=1}^{n} x_i = c$, then it is clear that $S(A_c, K, K_1)$ is empty and
 $(K, K_1) = A_c$.

further considerations we suppose that X

Theorem 4.2. If A is any non-empty subset of X and $K₁$ is every non-void subset of K, then *whenever for each* $p_a \in P$ *and there exists in the polar cone* K^0 *of* K such that $p_{\alpha}(a_0 - a) \leq x^*(a_0 - a) + \eta$, $\forall a \in A$.

Proof.

Let us suppose that, under the above hypotheses, $(K+K_1) \cap (a_0-A) \nsubseteq -(K+K_1)$, that is, there exists $a \in A$ so that $a_0 - a \in K + K_1 \setminus (-K - K_1)$. Then, $a_0 - a \neq 0$ and, because X is separated in Hausdorff's sense, there exists $p_a \in P$ such that $p_a (a_0 - a) > 0$. On the other hand, there exists $n \in N^*$ sufficiently large with $p_a(a_0 - a)/n \in (0,1)$ and the relation given by the hypothesis of theorem leads to $p_{\alpha}(a_0 - a) \le x^*(a_0 - a) + p_{\alpha}(a_0 - a)/n$ with $x^* \in K^0$ and $n \to \infty$, which implies that p_a ($a_0 - a$) ≤ 0 , a contradiction and the proof is completed. $(x_i) \in X : \sum_{i=1}^{n} x_i = c$, then it is clear that $S(A_i, K, K_1)$ is empty and
 $k_i, K, K_1 = A_c$.

If writher considerations we suppose that *X* is a Hausdorff locally convex space having

logigid dued by family $P = \{p_a : a \in I\}$ of ${}_{z}$, K , N , $j = a_{z}$.

If further considerations we suppose that X is a Hausdorff locally convex space having

logical dual space X'. In this framework, the next theorem contains a significant criterion

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e Prateo (minimal) efficient points, in particular, for the usual
this given cocount that the dual co uppose that *X* is a Hausdorff locally convex space having
 $p_a: \alpha \in I$ } of seminorms, ordered by a convex cone *K* and

framework, the next theorem contains a significant criterion

Pareto (minimal) efficient points, in p he topology induced by family $P = \{p_a : a \in I\}$ of seminorms, ordered by a convex cone K and
ts topological data space N' . In this framework, the next theorem contains a significant rreiterion
for the existence of the app

Remark 4.6. In general, the converse of this theorem is not valid at least in (partially) ordered separated locally convex spaces as we can see from the example considered in the previous remark for n=2. Indeed, if one assumes the contrary in the corresponding, mathematical's background, then, taking $\eta = \frac{1}{\tau}$ it follows that for each $\lambda_0 \in [0,1]$ there exists $c_1, c_2 \le 0$ such that Taking $\lambda_0 = \frac{1}{4}$ one obtains $|1 - 4\lambda| \le (c_1 - c_2)(1 - 4\lambda) + 1$, $\forall \lambda \in [0, 1]$ which for $\lambda = 0$ implies that $c_2 \leq c_1$ and for $\lambda = \frac{1}{2}$ leads to $c_1 \leq c_2$, that is, $|1-4\lambda| \leq 1$, $\forall \lambda \in [0,1]$, a contradiction. Let us to consider for each function $\varphi : P \to K^* \setminus \{0\}$ the convex cone $K_{\alpha} = \{x \in X : p(x) \le \varphi(p)(x), \forall p \in P\}$. The next theorem represents, in the more general context, a new important link between strong optimization and the approximative vector 4 *y non-empty subset of X and* K_1 *is every non-void subset of K, then*
 ever for each $p_a \in P$ *and there exists in the polar cone* $K^0 of K$
 $(a_0 - a) + \eta$, $\forall a \in A$.
 f the above hypotheses, $(K + K_1) \cap (a_0 - A) \not\subseteq$ ists $n \in N^*$ sufficiently large with $p_\alpha (a_0 - a)/n \in (0,1)$ and the repothesis of theorem leads to $p_\alpha (a_0 - a) \le x^* (a_0 - a) + p_\alpha (a_0 - a)/n$
 $\rightarrow \infty$, which implies that $p_\alpha (a_0 - a) \le 0$, a contradiction and the proof is
 emar *r* for each $p_a \in P$ and there exists in the polar cone $K^0 \circ f K$
 $(-a)+\eta, \forall a \in A$.

are above hypotheses, $(K+K_1) \cap (a_0 - A) \nsubseteq -(K+K_1)$, that is, there
 $\in K+K_1 \setminus (-K-K_1)$. Then, $a_0 - a \neq 0$ and, because X is separated

ex **Theorem 4.2.** *If A is any non-empty subset of X and K₁ is every non-void subset of K, then*
 c_a \in *eff* (A, K, K_1) whenever for each $p_a \in P$ and there exists in the polar cone K^b of K

uch that $p_a(a_a$ $P_a(a_0 - a) \le x^i (a_0 - a) + \eta$, $\forall a \in A$.

ppose that, under the above hypotheses, $(K + K_1)\cap (a_0 - A) \nsubseteq -(K + K_1)$, that is, there
 $\in A$ so that $a_0 - a \in K + K_1 \setminus (-K - K_1)$. Then, $a_0 - a \ne 0$ and, because X is separated
 $\text{for } S$ sen is suppose that, under the above hypotheses, $(K + K_1) \cap (a_0 - A) \nsubseteq -(K + K_1)$, tha
 $s \ a \in A$ so that $a_0 - a \in K + K_1 \setminus (-K - K_1)$. Then, $a_0 - a \neq 0$ and, because X is

ausdorff's sense, there exists $p_a \in P$ such that $p_a (a_0 - a) >$ 2 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ under the above hypotheses, $(K + K_1) \cap (a_0 - A) \nsubseteq -(K + K_1)$, that is, there $a_0 - a \in K + K_1 \setminus (-K - K_1)$. Then, $a_0 - a \neq 0$ and, because X is separated c, there exists $p_a \in P$ such that $p_a(a_0 - a) > 0$. On the other hand, there ex the convex cone 1, $a_0 - a \neq 0$ and, because *X* is separated
 $p_a (a_0 - a) > 0$. On the other hand, there
 $\in (0,1)$ and the relation given by the
 $-a) + p_a (a_0 - a)/n$ with $x^* \in K^0$ and

tion and the proof is completed.

is not valid at lea xists $a \in A$ so that $a_0 - a \in K + K$, $\langle -K - K \rangle$. Then, $a_0 - a \neq 0$ and, because X is separated
 Hausdoff''s senes, there exists $p_n \in P$ such that $p_n(a_n - a) > 0$. On the other hand, there
 K xists $n \in N^*$ sufficiently la

optimization together with its usual particular variant, respectively.

Theorem 4.3. If there exists $\varphi : P \to K^* \setminus \{0\}$ with then

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\n**Theorem 4.3.** If there exists
$$
\varphi: P \to K^* \setminus \{0\}
$$
 with then

\n
$$
eff(A, K, K_1) = \bigcup_{\substack{a \in A \\ \varphi \in P \to K^* \setminus \{0\}}} S\left(A \cap (a - K - K_1), K_\varphi\right)
$$
\nfor any non-empty subset K_1 of K .

\nProof

\nIf $a_0 \in \text{eff}(A, K, K_1)$ is an arbitrary element, then, in accordance with the point (i) of the

for any non-empty subset K_1 *of* K_2 .

Proof

If $a_0 \in \text{eff}(A, K, K_1)$ is an arbitrary element, then, in accordance with the point (i) of the Definition 4.1 and the hypothesis of the above theorem, we have for some $\varphi: P \to K^* \setminus \{0\}.$ *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 cm 4.3. *If there exists* $\varphi: P \to K^* \setminus \{0\}$ with then
 K, K_1 = $\bigcup_{\varphi \in P \to K^* \setminus \{0\}} S(A \cap (a - K - K_1), K_{\varphi})$
 non-empty subset K_1 *of K British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 orem 4.3. *<i>H* there exists $\varphi : P \to K^* \setminus \{0\}$ with then
 A, K, K , $) = \bigcup_{p \in P \to K^* \setminus \{0\}} S(A \cap (a - K - K_1), K_0)$
 A A K , $\varphi \in K$, $\varphi \notin K$, or $\frac{8 \text{ } \textcircled{1}}{2}$
 $\frac{8 \text{ } \textcircled{2}}{2}$
 $\frac{1}{2}$ *British Journal of Mahematics & Computer Science 4(8), 1048-1073, 2014*
 f there exists $\varphi: P \to K^{\times} \setminus \{0\}$ *with then*
 $\cdot \bigcup_{a \in A} S\{A \cap (a - K - K_1), K_{\varphi}\}$
 pty subset K_1 of K .

(*A,K, K₁*) is an arbitrary e of K.

an arbitrary element, then, in accordance with the point (i) of the

the hypothesis of the above theorem, we have
 $K + K_1 \subseteq K \subseteq K_\varphi$ for some $\varphi : P \to K^* \setminus \{0\}$.
 $a_0 - K - K_1$, K_φ).
 $K_\varphi = K_\varphi + K_1 + K_\varphi$.
 K_\var rbitrary element, then, in accordance with the poi

by hypothesis of the above theorem,
 $K_1 \subseteq K \subseteq K_\varphi$ for some $\varphi : P \to K^* \setminus \{0\}$.
 $[-K_1), K_\varphi$.
 $\bigcup_{\substack{a \in A \\ P \to K^* \setminus \{0\}}} S(A \cap (a_0 - K - K_1), K_\varphi)$. Conversely,

for at British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014

: $P \rightarrow K^* \setminus \{0\}$ with then
 $\bigcap (a - K - K_1), K_{\varphi}$

a arbitrary element, then, in accordance with the point (i) of the

the hypothesis of the above

Therefore, $a_0 \in S(A \cap (a_0 - K - K_1), K_\varphi)$.

Hence,
$$
eff(A, K, K_1) \subseteq \bigcup_{\substack{a \in A \\ \varphi: P \to K^*\setminus \{0\}}} S(A \cap (a_0 - K - K_1), K_\varphi)
$$
. Conversely, let now

for at least one elements $a_0 \in A$ and $\varphi : P \to K^* \setminus \{0\}$. Then, $a_1 \in A \cap (a_0 - K - K_1)$ and $A \cap (a_0 - K - K_1) - a_1 \subseteq K_{\alpha}$, that is, $p(a-a_1) \le \varphi(p)(a-a_1), \forall a \in A \cap (a_0 - K - K_1), \ p \in P$ which implies immediately that $p(a_1-a) \leq -\varphi(p)(a_1-a)+\eta$, $\forall a \in A \cap (a_0-K-K_1)$, $p \in P, \eta \in (0,1)$ and, by virtue of Theorem 2.2 one obtains $a_i \in \text{eff}(A \cap (a_0 - K - K_1), K, K_1)$. But eff $(A \cap (a_0 - K - K_1), K, K_1) \subseteq$ eff (A, K, K_1) . British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014
exists $\varphi : P \rightarrow K^* \setminus \{0\}$ with then
 $S\{A \cap (a - K - K_1), K_{\varphi}\}$
(et K_1 of K .
(et K_1 of K .
(et K_1 of K .
(et K_1 of K .
(et K_1 of *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 there exists $\varphi: P \rightarrow K^* \setminus \{0\}$ with then
 $\bigcup_{a \in A} S(A \cap (a - K - K_1), K_0)$
 y subset K_1 *of K*.
 A,K,K₁) is an arbitrary element, then, *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 orcen 4.3. *If there exists* $\varphi : P \rightarrow K^* \setminus \{0\}$ *with then*
 A, K, K_1 = $\bigcup_{\varphi \in \mathbb{R}^{K_0 \setminus \{0\}} \setminus \{0\}} S(A \cap (a - K - K_1), K_{\varphi})$
 arrow in the British Journal of Mathematics & Computer Science 4(5), 1948-1075, 2014
 $D: P \rightarrow K^* \setminus \{0\}$ with then
 $\bigcap (a - K - K_1), K_{\varphi}$

of K.

an arbitrary element, then, in accordance with the point (i) of the

the hypothesis of Find the point (i) of the
theorem, we have
interesting the state of the
theorem, we have
 $\varphi: P \to K^* \setminus \{0\}$. Then,
it is immediately that
by virtue of Theorem *a A a K K* 1 0 1 *A a K K a K* 0 1 1 , *p a a p a a a A a K K* 1 1 0 1 , , *p P p a a p a a* 1 1 , *a A a K K* 0 1, *p P* , 0,1 *a* $Q(A \cap (A \cap K, K))$
 a $P(A \cap (A \cap K - K_1), K_0)$
 abset K_1 *af* A *f* K_1 *a a* for the above theorem, we have
 and the hypothesis of the above theorem, we have
 $Q = K + K_1 \subseteq K \subseteq K \subseteq K_p$ for some $\varphi: P \to K^* \setminus \{0\}$. *non-empty subset* K_1 of K .
 $a_5 \in \text{eff}(A, K, K_1)$ is an arbitrary element, then, in accordance with the point (i) of the $\text{min}(A_1, A_2)$ and the hypothesis of the above theorem, we have $\left((a_0 - K - K_1) - a_0 \leq K + K_1 \leq K$ *f* $\mathfrak{g}_{\mathbb{R}}$ is an arbitrary element, then, in accordance with the point (i) of the
and the hypothesis of the above theorem, we have
 $\mathfrak{g}_{\mathbb{R}} \in K + K_1 \subseteq K \subseteq K_{\mathbb{R}}$ for some $\varphi: P \to K^* \setminus \{0\}$.
 $\Gamma(\mathfrak{a}_0 - K$ *h* $a_i \in \text{eff}(A, K, K_i)$ is an arbitrary element, then, in accordance with the point (i) of the

befinition 4.1 and the hypothesis of the above theorem, we have
 $A \cap (a_0 - K - K_1) - a_0 \subseteq K + K_1 \subseteq K \subseteq K_\phi$ for some $\varphi: P \to K^* \setminus \{$ E_K for some $\varphi: P \to K^* \setminus \{0\}$.
 K_{φ}).
 $S(A \cap (a_0 - K - K_1), K_{\varphi})$ Conversely, let now

east one elements $a_0 \in A$ and $\varphi: P \to K^* \setminus \{0\}$. Then,
 $K - K_1) - a_1 \subseteq K_{\varphi}$, that is,
 $a_0 - K - K_1$, $p \in P$ which implies i *K* - *K*₁), *K*_{*g*}).
 $\int_{\mathbb{R}^{d}} S(A \cap (a_{0} - K - K_{1}), K_{\varphi})$. Conversely, let now

for at least one elements $a_{0} \in A$ and $\varphi : P \rightarrow K^{\dagger} \setminus \{0\}$. Then,
 $i \cap (a_{0} - K - K_{1}) - a_{1} \subseteq K_{\varphi}$, that is,
 $i \in A \cap (a_{0} - K - K_{1})$, nce, $\operatorname{eff}(A, K, K_1) \subseteq \bigcup_{\sigma \in \mathcal{F}_n K^+ \setminus \{0\}} S\{A \cap (a_0 - K - K_1), K_\phi\}$ Conversely, let now
 $\in S\{A \cap (a_0 - K - K_1), K_\phi\}$ for at least one elements $a_0 \in A$ and $\varphi : P \to K^+ \setminus \{0\}$. Then,
 $\in A \cap (a_0 - K - K_1)$ and $A \cap (a_0 - K$ Hence, $\text{erf}(A, K, K_1) \subseteq \bigcup_{\mathbf{c} \in \mathbb{R}^n \times \mathbb{R}^n} S(A \cap (a_n - K - K_1), K_\phi)$ Conversely, let now
 $a_i \in S(A \cap (a_0 - K - K_1), K_\phi)$ for at least one elements $a_0 \in A$ and $\phi: P \to K^* \setminus \{0\}$. Then,
 $a_i \in A \cap (a_0 - K - K_1)$ and $A \cap (a_$ eff $(A, K, K_1) \subseteq \bigcup_{\substack{e, p, n \in K^* \setminus \{0\}}} \{A \cap (a_0 - K - K_1), K_{\varphi}\}$ Conversely, let now
 $\int_{e, p, n \in K^* \setminus \{0\}} \int_{\mathbb{R}^n} \{A \cap (a_0 - K - K_1), K_{\varphi}\}$ Conversely, let now
 $\int_{a_0 - K - K_1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{A \cap (a_0 - K - K_1)$ K_{φ}) for at least one elements $a_0 \in A$ and $\varphi : P \rightarrow K^* \setminus \{0\}$. Then,

nd $A \cap (a_0 - K - K_1) - a_1 \subseteq K_{\varphi}$, that is,
 $\forall x \in A \cap (a_0 - K - K_1), \quad p \in P$ which implies immediately that
 $+ \eta, \forall a \in A \cap (a_0 - K - K_1), \quad p \in P, \eta \in (0,1)$ *a*₁ e *S*{ $A \cap (a_0 - K - K_1)$, K_o } for at least one elements $a_0 \in A$ and $\phi: P \rightarrow K^* \setminus \{0\}$. Then,
 $a_i \in A \cap (a_0 - K - K_i)$ and $A \cap (a_0 - K - K_1) - a_i \subseteq K_o$, that is,
 $p(a - a_i) \leq \phi(p)(a - a_i)$, $\forall a \in A \cap (a_0 - K - K_i)$, $p \in P$ which i *a*₁) ≤ *φ*(*p*)(*a* -*a*₁), ∀*a* ∈ *A* \cap (*a*₀ - *K* - *K*₁), *p* ∈ *P* which implies immediately that -*a*) ≤ *-φ*(*p*)(*a*₁-*a*)+*n*, ∀*a* ∈ *A* \cap (*a*_q-*K*-*K*₁), *p* ∈ *P*, *n* ∈ (0,1) and, by virtu

Indeed, for any $t \in eff(A \cap (a_0 - K - K_1), K, K_1)$ and $h \in A \cap (t - K - K_1)$ we have $h \in A \cap (a_0 - K - K_1) \cap (t - K - K_1) \subseteq t + K + K_1$ that is, $A \cap (t - K - K_1) \subseteq t + K + K_1$ and by point (i) of Definition 2.1 one obtains $t \in eff(A, K, K_1)$. This completes the proof.

Remark 4.7. The hypothesis $K \subseteq K_{\varphi}$ imposed upon the convex cone K is automatically satisfied whenever K is a supernormal (nuclear) cone and it was used only to prove the inclusion $(A,K,K_1) \subseteq \bigcup S\big(A\cap(a-K-K_1),K_{\varphi}\big)$. When K is any pointed convex cone, A is a $\{0\}$ bbtains $a_1 \in eff(A \cap (a_0 - K - K_1), K, K_1)$.
 $A \cap (a_0 - K - K_1), K, K_1) \subseteq eff(A, K, K_1)$.

for any $t \in eff(A \cap (a_0 - K - K_1), K, K_1)$ and $h \in A \cap (t - K - K_1)$
 $(a_0 - K - K_1) \cap (t - K - K_1) \subseteq t + K + K_1$, that is, $A \cap (t - K - K_1) \subseteq t + K + t$

of Definition 2.1 one $t_0 - K - K_1$, K , K_1) \subseteq *eff* (A, K, K_1) .

any $t \in eff$ $(A \cap (a_0 - K - K_1)$, K , K_1) and $h \in K - K_1$ \cap $(t - K - K_1) \subseteq t + K + K_1$, that is, $A \cap (t -$

finition 2.1 one obtains $t \in eff$ (A, K, K_1) . This com

e hypothesis $K \subseteq$

non-empty subset of *X* and $a_0 \in \text{eff}(A, K)$, then, by virtue of (i) in Definition 2.3, it follows that that is, $A \cap (a_0 - K) - a_0 = \{0\} \subset K_\varphi$. Hence, $a_0 \in S(A \cap (a_0 - K), K_\varphi)$ for every mapping φ : $P: K^* \setminus \{0\}$ and the next corollary is valid.

Corollary 4.3.1. *For every non-empty subset A of any Hausdorff locally convex space ordered by an arbitrary, pointed convex cone K with its dual cone* K^* *we have*

$$
eff(A,K) = \bigcup_{\substack{a \in A \\ \varphi: P \to K^* \setminus \{0\}}} S\big(A \cap (a-K), K_{\varphi}\big)
$$

Remark 4.8. Clearly, the announced theorem represents a significant result concerning the possibilities of scalarization for the study of Pareto efficiency in separated locally convex spaces, as we can see also for the particular cases of Hausdorff locally convex spaces ordered by closed, pointed and normal cones. The above coincidence between the equilibria points sets and Choquet British Journal of Mathematics & Com,
 $\bigcup_{a \in A} S\Big(A\cap (a-K), K_{\varphi}\Big)$ $\text{Clearly, the announced theorem represents a si
f scalarization for the study of Pareto efficiency in}$ British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014

, K) = $\bigcup_{a \in A^2 \to A^2 \setminus \{0\}} S\left(A \cap (a - K), K_{\varphi}\right)$

rk 4.8. Clearly, the announced theorem represents a significant result concerning the

rk 4.8 *Pritish Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 Pff $(A, K) = \bigcup_{\substack{a \neq a \neq 4 \\ \text{if } a \neq 4}} S(A \cap (a - K), K_{\varphi})$
 Remark 4.8. Clearly, the announced theorem represents a significant result concerni *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
= $\bigcup_{\substack{a \in A \\ \varphi: P \to K^* \setminus \{0\}}} S(A \cap (a - K), K_{\varphi})$
1.8. Clearly, the announced theorem represents a significant result concerning the sof scalari boundaries togeher with its immediate corrolaries remains valid if one replaces ε by K_1 .

Finally, following $[1]$ and $[2]$, we present two studies concerning the eco – efficiency which were not used yet.

5 Some Applications

5.1 The Equilibria of Multidimensional Ecosystems

This part of the present section is devoted to some new considerations for the study of the equilibrium points in any "usual" multidimensional ecosystem through the agency of the corresponding interdependences assumed between its parties, in the optimal background offered by the natural resources and under the ecological constraints and conversely. The equilibrium points are examined in the context given by a suitable sustainable development of ecosystem, based on the dynamic interactions between its components, in particular, the ecological and the social phenomenons and processes and more, the natural resources and the human needs, respectively. Thus, the foundations of this research paper consist in some new and very possible critical original ideas. Thus, first of all, we consider the useful modern concept of "sustainable development" and we propose its new corresponding extension concerning the general optimal integration through the agency of equilibrium for all the component parties in an every multidimensional ecosystem. The more general abstract efficiency has its immediate applications in the study of the equilibrium for multidimensional ecosystems because we think that our vectorial and, consequently, strong notion named *over all sustainable dimensionality* is the main consequence and generates the individual competitive and the mutual *sustainable targeting.* Our spline functions (splines) introduced in H-locally convex spaces backgrounds with their proper optimal interpolation properties applied in the corresponding particular cases of cardinal splines involved for the long – time decision – making processes whenever it is known their previous behaviours in the field, with appropriate regularities and one asks their evolution in the immediate future on a convenient time period. One of the reasons for this approach is our opinion that in any best (optimal) context of real (natural) sustainability the interactions between the ecological systems and the dynamics of the economical and social processes must run properly having the maximum degrees of regularity. Finally, we formulate some open problems and we give adequate concluding remarks. Concerning the sustainability and the equilibria, by our opinion, one of the way to examine the multidimensional ecosystems is the study of the interactions between their components based on the tendency by the individual objectives targeting continued by the corresponding generalization with the mutuals and afterwards the multiple sustainable targeting which, through the agency of the convergence for the interests, lead to the optimal solution (the equilibrium) named by us the *over all sustainable dimensionality.* Starting from the analysis of the existence and the behavior of the solutions which generate the equilibrium points for an ecosystem

with two conflicting component parties under adequate hypotheses and following the modelling sustainable use of natural resources with applications by simulation to fishery and forestry, here we shall try to unify some of the basic results of the above research papers and so on, offering a proposal of generalization.

Let us consider an ecosystem in $n \ge 2$ conflicting parties $X_i(i = \overline{1,n})$. If we denote by $f_i(t, x_i, x_i)$ $(i, j = \overline{I, n})$ the growth rate of the quality of life for X_i with respect to X_j , then in partial accordance wih the usual agreements and following our ideas the mutual interactions witch lead to the equilibrium points, amongst X_i and X_j can be described explicitly in many cases by the following kind of general differential equations system: **Example 10**
 Example 10 Example 10 *x following the Computer Science 4(8), 1048-1073, 2014***

conflicting component parties under adequate hypotheses and following the modelling

ie use of ratural resources with app** *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 ig component parties under adequate hypotheses and following the modelling
 interactial cesoures with applications by simulation to fishery and

(1)
$$
\dot{x}_i = f_i(t, x_j, x_i)x_i - r_{ij}(t), i, j = \overline{1, n}
$$

where x_i represents the level of life's quality for X_i , $r_{ij}(t)$ means the time-dependent harvest of the interaction $X_i \leftrightarrow X_j$ and f_i $(i = 1, n)$ satisfies proper conditions (see, for example, the particular bidimensional cases and the unidimensional basic models in the economic theory of resource management).

Whenever for $i = \overline{1,n}$ there exists x_i^* solution in (1) such $x_i^* \le x_i$ for all possible x_i and $t \in [t_0, t_i)$ one says that x_i^* is a *solution* to achive the *individual competitive targeting*; if $(t) \leq x_i(t)$ $\begin{cases} x_i(t) \le x_i(t) \\ x_j^*(t) \le x_j(t) \end{cases}$ ($i \ne j$) for all admissible x_i and x_j and $t \in [t_0, t_0]$ $\begin{cases} \begin{array}{c} i & i \\ \ast & i \end{array} \end{cases}$ $(i \neq j)$ for all admissible $\left(x_i^*(t) \leq x_i(t)\right)$ $\leq x_i(t)$ $\leq x_i(t)$ $x_i^*(t) \le x_i(t)$ (1) $x_i^*(t) \le x_i(t)$ (1) $j \mathcal{U}$ $j \rightharpoonup \mathcal{N}_j \mathcal{U}$ $i \left(i \right) = \lambda_i \left(i \right) \quad \left(\dots \right)$ $^{*}(t)$ \leq $\frac{1}{2}$ (t) $(1, 1)$ $(2, 1)$ $f(t) < r(t)$ for all admissible x_i and x_j and $t \in [t_0, t_{ij}]$, then we say that we have a *mutual sustainable development* based on a *reciprocal sustainable targeting*. Obviously, if $x_i^*(t) \le x_i(t)$ for all possible allowed values of x_i , $i = \overline{I}$, *n* and *t* sufficiently large, then it is about of *the strong sustainable development* grounded on *the corresponding global sustainable targeting*. If there exists a feasible solution \widetilde{x}_i , $(i = \overline{I,n})$ of (1) such that it is not possible to have $(x_1(t), x_2(t),..., x_n(t)) < (\widetilde{x}_1(t), \widetilde{x}_2(t),..., \widetilde{x}_n(t))$ for *t* sufficiently vast, then we call $(\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_n)$ the vector sustainable equilibrium for (1). We must specify that all these concepts are introduced here under the main restriction that, from the dynamical point o view, the equilibrium was considered being moulded on every cyclicity submitted at least to the approximate periodical states of system's life.

Let us consider as a pertinent example the following bidimensional case offered by an ecosystem made up of two parties X_1 and X_2 . Suppose that unit X_1 represents the pollutive party (in our case the power plant) and it will upset the vital equilibrium of an environmental system X_2 of high scenic interest, measured with reference to the quality life of the system itself. In our case, since it is extremely significant to know how systems develop in time, let's consider the case as a

Example 10 British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014

non autonomous one. Let $f_2(t, x_1, x_2)$ be the growth rate of the quality of life of system X_2

(measured by a quantity inversely pr **Example 19** British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014

non autonomous one. Let $f_2(t, x_1, x_2)$ be the growth rate of the quality of life of system X_2

(measured by a quantity inversely pr growth rate of system X_1 (measured by the kwh generated).

Since the building up of a model does not consist only in specifying the mathematical formulation which holds it, but also in defining the hypothesis on which its existence is based, let's suppose that the interdependance between the conflicting systems be such as to satisfy the four following conditions:

(i) In the absence of system X_1 , the level of the quality of life for system X_2 is restored

$$
X_2' = a_2
$$
.

British Journal of Mathematics & Computer Science 4(8), 1043-1073, 2014
 about the rate *a* **flow in a gradient of the growth rate of the quality of life of system** X_2 **

d by a quantity inversely proportional to the pa** We are implying that for system X_2 the problem arises from the its proximity to system X_1 and whenever the latter is missing, the former would resume growth up to a level which would allow full use of natural available resources, by the disappearence of pollutive particulates. The term a_2 represents the growth coefficient of the quality of life or, otherwise, the reduction of the pollution level. Such a value is positive since in the absence of the pollutive cone, which runs over the specific area, the natural course would resume exponentially by integral curve. d by a quantity inversely proportional to the particulate fall-out) and $f_1(t, x_1, x_2)$ the te of system X_1 (measured by the kwh generated).

building up of a model does not consist only in specifying the mathematical building up of a model does not consist only in specifying the mathematical formulation
builds it, but also in defining the hypothesis on which its existence is based, let's suppose
terdependance between the confliciting

$$
x_2(t) = x_2^0 e^{a_2 t}.
$$

(ii) In the absence of the environmental system X_2 , system X_1 will develop its activity, by a logistic growth up to the level of maximum productive capacity, that is,

$$
x_1' = a_1 x_1 - a_{11} x_1^2
$$

with asymptote $\frac{a_1}{a_{11}}$. We are admitting, in conclusion, that for system X_1 the problem lies in the proximity of system X_2 and, therefore, in the absence of the latter, the former will perform its activity up to the maximum level $\frac{a_1}{a_1}$, with serious damages to the environment by integral curve We are admitting, in conclusion, that for system X_1 the prob
 X_2 and, therefore, in the absence of the latter, the former will

um level $\begin{pmatrix} a_1 \\ a_1 \end{pmatrix}$, with serious damages to the environment
 $\begin{pmatrix} a_1 \\ a_1$ e are admitting, in conclusion, that for system X_1 the

² and, therefore, in the absence of the latter, the former

m level $\begin{array}{c} a_1 \\ a_1 \end{array}$, with serious damages to the environm
 $\begin{array}{c} a_1 \\ e^{-a_1 t} + a_{11} \end{array}$ f natural available resources, by the disappearence of pollutive particulates. The term a_2 s the growth coefficient of the quality of life or, otherwise, the reduction of the pollution fit a value is positive since in beflicient of the quality of life or, otherwise, the reduction of the pollution
ositive since in the absence of the pollutive cone, which runs over the
course would resume exponentially by integral curve.

accourse woul er is missing, the former would resume growth up to a level which would allow
available resucuses, by the disappearence of pollutive particulates. The term a_2
available resucuses, by the disappearence of pollutive part a namone cosources, vy a casappearate to pointary the reduction of the pollution
with coefficient of the quality of life or, otherwise, the reduction of the pollution
use is positive since in the absence of the pollutive $x_2(t) = x_2^0 e^{\alpha y}$.

(ii) In the absence of the environmental system X_1 , system X_1 will develop its activity,

by a logistic growth up to the level of maximum productive capacity, that is,
 $x_1' = a_1 x_1 - a_1 x_1'^2$

$$
x_1(t) = \frac{a_1}{\left(\frac{a_1}{x_0} - a_{11}\right) e^{-a_1 t} + a_{11}}
$$

the differential equation and the set P, structured by these projections, is called *space of the phases*

of the equation; the arrows shows the direction by which the projected point draws its orbit by the increase of *t* (Fig. 1).

Fig. 1. The orbit by the increase of *t.*

(iii) The activity of system X_1 determines a reduction of the growth rate of the quality of life for system X_2 at a rate proportional to that activity and, more precisely, to the size and physical and chemical characteristics of the cone, which runs over the system X_2 ; let's suppose in this situation that the growth rate moves from a_2 to $(a_2 - a_{21}x_1)$, so t
 a

ate of the quality of

recisely, to the size

ver the system X_2 ;

to $(a_2 - a_{21}x_1)$, so that:

$$
X'_{2} = (a_{2} - a_{21}x_{1})x_{2}
$$

$$
\frac{\partial f_{2}}{\partial x_{1}} < 0 = -a_{21}
$$

On the meaning of coefficient a_{21} we assume that the interaction between the two systems is proportional both to X_1 and to X_2 , $\alpha(t)$, x_1 , x_2 , with $\alpha(t)$ proportionally constant, as a measure **Fig. 1. The orbit by the increase of t.**
 Fig. 1. The orbit by the increase of t.
 X_2 at a rate proportional to that activity and, more precisely, to the size

d chemical characteristics of the cone, which runs over of lower intensity of interaction. A fraction of this interaction turns out in the mitigation of the activity X_1 , and it reduces the ill effects on the system X_2 ; as a first approximation, one could suppose that the size of this reduction is proportional to the intensity of the interaction at work. **Fig. 1. The orbit by the increase of** *t***.**
(iii) The activity of system X₁ determines a reduction of the growth rate of the quality of
life for system X₂ at a rate proportional to that activity and, more precisely, t Specifying such a fraction as $\beta \alpha(t) x_1 x_2$, where β measures the capacity of X_2 to protect from X_1 , and since this is not possible, we can say that it measures the possibility by X_1 to mitigate its action over X_2 , that is,

*a*₂₁ = $\beta \alpha(t)$.
 *a*₂₁ = $\beta \alpha(t)$.

(iv) The reduction of the quality of life in system X_2 determines a decrease in the production rate of X_1 in proportion to the size of the reduction. Suppose that such a pro (iv) The reduction of the quality of life in system X_2 determines a decrease in the production rate of X_1 in proportion to the size of the reduction. Suppose that such a **British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014**
 (x) The reduction of the quality of life in system X_2 determines a decrease in the

production rate of X_1 in proportion to the size of the *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

(*t*).

(*x*) The reduction of the quality of life in system X_2 determines a decrease in the

production rate of X_1 in proportion to the siz *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

(*t*).

(*t*)

(*t*)

(*x*) The reduction of the quality of life in system X_2 determines a decrease in the

roduction rate of X_1 in proporti Mathematics & Computer Science 4(8), 1048-1073, 2014

life in system X_2 determines a decrease in the

to the size of the reduction. Suppose that such a

ges from the value $(a_1 - a_1, x_2)$ to the value

mes:
 $\langle 0 = -a_{12$

$$
x_1' = (a_1 - a_{11}x_1 - a_{12}x_2)x_1
$$
 with $\frac{\partial f_1}{\partial x_2} < 0 = -a_{12}$ (Fig. 2)

Fig. 2. The dependence between X_1 and X_2

The coefficient a_{12} measures the disadvantage of system X_1 because of interaction with system X_2 : particulate fall-out creates unpopular feelings towards the Electricity Board and, by production rate of X_1 in proportion to the size of the reduction. Suppose that such a
production rate $f_1(I, x_1, x_2)$ changes from the value $(a_i - a_{i1}, x_2)$ to the value
 $(a_i - a_{i1}, x_1 - a_{i2}, x_2)$, so that it becomes:
 x'_1 specifying $a_{12} = \gamma a_{12}$, we can interpret γ as a measure of the pollution rate which forces X_1 to a decrease in its activity: $\alpha_1 - a_{11} x_1 - a_{22} x_2$), so that it becomes:
 $x'_1 = (a_1 - a_{11}x_1 - a_{12}x_2)x_1$ with $\frac{\partial f_1}{\partial x_2} < 0 = -a_{12}$ (Fig. 2)
 B.
 A
 A
 a
 **Fig. 2. The dependence between
** x_1 **and** x_2 **

Fig. 2. The dependence betw** μ_1 measures the disadvantage of system X_1 because of interaction with sy
fall-out creates unpopular feelings towards the Electricity Board and
 αa_{12} , we can interpret γ as a measure of the pollution rate whi

$$
a_{21} = \gamma \beta \alpha(t).
$$

In conclusion the interaction between the two systems is ruled by the following quadratic model, an equation system shaped as:

(2)
$$
\begin{cases} x_1' = (a_1 - a_{11}x_1 - a_{12}x_2)x_1 \\ x_2' = (a_2 - a_{21}x_1)x_2 \end{cases}
$$

x₁ **c**
 x₁ **c**
 x₁ **c**
 x₁₂ **c**
 z₁₂ **c**
 z z₁₂ **c**
 x z z *x* **c**
 z z *x* **c**
 x z *x* **c**
 x The a_{11} coefficient can be called intersystem interaction coefficient, whilst the terms a_{12} and a_{21} can be called intrasystem interaction coefficients. The latter, moreover, could be envisaged as Fig. 2. The dependence between
 X_1 and X_2

The coefficient a_{12} measures the disadvantage of system X_1 because of interaction with system
 X_2 : particulate fall-out creates upoppular feelings towards the Ele interaction or interference between the two systems, whilst d_{12} and d_{21} measure the damage that such interaction involves for each system.

Having set out the problem in such a way, to study the interaction in the short run, we now proceed to discover, if there exist, the equilibrium solutions: a strong pollution at present, due also to adverse weather conditions, requires the search for a solution to be chosen amongst the feasible ones, such as to provoke an immediate reduction of the ill effects determined by the level of activity of X_1 . *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

ones, such as to provoke an immediate reduction of the ill effects determined by the level of

activity of X_1 .

Defining D the determinant of t ics & Computer Science 4(8), 1048-1073, 2014

he ill effects determined by the level of

on in the above system which can be also
 $a_{12}a_{21}$ e con
 $a_{12}a_{21}$,

Defining *D* the determinant of the matrix of the equation in the above system which can be also

British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014\n\nones, such as to prove the an immediate reduction of the ill effects determined by the level of\n\ncativity of
$$
X_1
$$
.\n\nDefining *D* the determinant of the matrix of the equation in the above system which can be also\ncalled *an ecosystem matrix*, that is,\n
$$
D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{vmatrix} = -a_{12}a_{21} \text{ e} \text{ con}
$$
\nand by considering\n
$$
H_1 = \begin{vmatrix} a_1 & a_1 \\ a_2 & 0 \end{vmatrix} = -a_2a_{12},
$$
\n
$$
H_2 = \begin{vmatrix} a_{11} & a_1 \\ a_2 & 0 \end{vmatrix} = a_2a_{11} - a_1a_{21}
$$
\nthen the equilibrium points of the mentioned equations system result as follows in terms of the following points:\n
$$
E_0: x_1 = 0, x_2 = 0;
$$
\n
$$
E_1: x_1 = \frac{a_1}{a_1}, x_2 = 0
$$
\n
$$
E_2: x_1 = \frac{a_2}{a_2} = \frac{H}{D}, x_2 = \frac{a_1a_{21} - a_1a_2}{a_{12}a_{21}} = \frac{H_2}{D}
$$
\n\nBy the usual analytic point of view the point E_1 is the intersection of straight line $x_1 = \frac{a_1}{a_{11}}$ with the axis x_1 ($x_2 = 0$) and the point E_2 is the intersection between the lines

then the equilibrium points of the mentioned equations system result as follows in terms of the

British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014
ness, such as to proveke an immediate reduction of the ill effects determined by the level of
activity of
$$
X_1
$$
.
Defining *D* the determinant of the matrix of the equation in the above system which can be also
alled *an ecosystem matrix*, that is,

$$
D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{vmatrix} = -a_{12}a_{21} \text{ e con}
$$
and by considering

$$
H_1 = \begin{vmatrix} a_1 & a_1 \\ a_2 & 0 \end{vmatrix} = -a_2a_{12},
$$

$$
H_2 = \begin{vmatrix} a_{11} & a_1 \\ a_{21} & a_2 \end{vmatrix} = a_2a_{11} - a_1a_{21}
$$
Then the equilibrium points of the mentioned equations system result as follows in terms of the
following points:

$$
E_0 : x_1 = 0, x_2 = 0;
$$

$$
E_1 : x_1 = \frac{a_1}{a_{11}}, x_2 = 0
$$

$$
E_2 : x_1 = \frac{a_2}{a_{21}} = \frac{H}{D}, x_2 = \frac{a_1a_{21} - a_1a_2}{a_{12}a_{21}} = \frac{H_2}{D}
$$
By the usual analytic point of view the point E_1 is the intersection of straight line $x_1 = \frac{a_1}{a_{11}}$ with
he axis x_1 ($x_2 = 0$) and the point E_2 is the intersection between the lines

$$
\begin{vmatrix} \vdots x_1 - \frac{a_2}{a_{21}} = 0 & \text{and } r_2 : x_2 = \frac{a_1}{a_{12}} = \frac{a_1}{a_{21}} x_1 \\ a_2 = \frac{a_1}{a_{21}} = \frac{a_1}{b_1} x_1 \end{vmatrix}
$$

By the usual analytic point of view the point E_1 is the intersection of straight line $x_1 = \frac{a_1}{x_1}$ with $x_1 = \frac{a_1}{a_{11}}$ with

the axis x_1 ($x_2 = 0$) and the point E_2 is the intersection between the lines

$$
r_1: x_1 - \frac{a_2}{a_{21}} = 0
$$
 ad $r_2: x_2 = \frac{a_1}{a_{12}} - \frac{a_{11}}{a_{12}} x_1$

and by considering
 $H_1 = \begin{vmatrix} a_1 & a_1 \\ a_2 & 0 \end{vmatrix} = -a_2 a_1_2$,
 $H_2 = \begin{vmatrix} a_1 & a_1 \\ a_2 & 0 \end{vmatrix} = a_2 a_1_1 - a_1 a_2_1$

or the equilibrium points of the mentioned equations system result as follows in terms of the

llowing p and by considering
 $H_1 = \begin{vmatrix} a_1 & a_1 \ a_2 & 0 \end{vmatrix} = -a_2a_{12}$,
 $H_2 = \begin{vmatrix} a_1 & a_1 \ a_2 & 0 \end{vmatrix} = a_2a_{11} - a_1a_{21}$

then the equilibrium points of the mentioned equations system result as follows in terms

following poin and by considering
 $H_1 = \begin{vmatrix} a_1 & a_2 \\ a_2 & 0 \end{vmatrix} = -a_2 a_1$,
 $H_2 = \begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = a_2 a_{11} - a_1 a_{21}$

points of the mentioned equations system result as follows in terms of the
 $E_0: x_1 = 0, x_2 = 0;$
 $E_1: x_$ $L = \begin{vmatrix} a_{21} & 0 \\ a_{22} & 0 \end{vmatrix} = -a_{12}a_{21}$ e com

and by considering
 $H_1 = \begin{vmatrix} a_1 & a_1 \\ a_2 & 0 \end{vmatrix} = -a_2a_{22}$,
 $H_2 = \begin{vmatrix} a_{11} & a_1 \\ a_{21} & a_2 \end{vmatrix} = a_2a_{11} - a_1a_{21}$

so f the mentioned equations system result as geometric locus of the points where respectively x'_2 and x'_1 turn to null, beyond points on the axis $H_3 = \begin{vmatrix} a_1 & a_2 \\ a_{21} & a_2 \end{vmatrix} = a_2a_{11} - a_1a_{31}$
 x $F_0: x_i = 0, x_2 = 0;$
 $F_0: x_i = 0, x_3 = 0;$
 $F_1: x_i = \frac{a_i}{a_{11}}, x_2 = 0$
 $F_2: x_i = \frac{a_2}{a_{11}} = \frac{H}{D}, x_2 = \frac{a_1a_{21} - a_1a_2}{a_{12}a_{21}} = \frac{H_2}{D}$

By the usual analy derivatives: positive in the area in which it is concordant with the respective axis and negative in the area in which it is discordant, Fig. 3a and 3b. On the very straight lines, by definition, the orbit of the system by crossing them will become horizontally tangent on $r₁$ and vertically tangent on r_2 . The point E_2 has always a positive abscissa, given that H_1 e D are both negative, whilst *F_n* : $x_i = 0$, $x_2 = 0$;
 F₁ : $x_i = \frac{a_i}{a_{i+1}}$, $x_2 = 0$
 F₂ : $x_i = \frac{a_2}{a_{i+2}} = \frac{H}{D}$, $x_2 = \frac{a_1a_{21} - a_1a_2}{a_{12}a_{21}} = \frac{H_1}{D}$
 By the usual analytic point of view the point *F₁* is the intersec and x'_1 turn to null, beyond points on the axis
the other one. The arrows show the signs of the
ordant with the respective axis and negative in
n the very straight lines, by definition, the orbit
ontally tangent on r_1 $x_2 = 0$
 $\frac{H}{D}$, $x_2 = \frac{a_1 a_{21} - a_{11} a_2}{a_{12} a_{21}} = \frac{H_2}{D}$

is the intersection of straight line $x_1 = \frac{a_1}{a_{11}}$ with

ection between the lines
 x'_2 and x'_1 turn to null, beyond points on the axis

the ot $x_2 = 0$
 $\frac{H}{D}$, $x_2 = \frac{a_1 a_{21} - a_{11} a_2}{a_{12} a_{21}} = \frac{H_2}{D}$

is the intersection of straight line $x_1 = \frac{a_1}{a_{11}}$ with

section between the lines
 x'_2 and x'_1 turn to null, beyond points on the axis

the o

$$
x_2 \ge 0
$$
 according to $H_2 = a_2 a_{11} - a_1 a_{21} \le 0$, i.e. $\frac{a_2}{a_{21}} \le \frac{a_1}{a_{11}}$:

Fig. 3. Signs of the derivaties.

For a real understanding of the system considered, we are only interested in the quadrants where of life.

Fig. 4. Real understanding iof the system.

In Fig. 4 are drawn generic straight lines as the only way in which they can be set in the plane from a realistic point of view because of the sign of H_2 , and the likely equilibrium points. Those capacity of the environment for system X_1 (maximum yield in production corresponding to the asymptote of the logistic in accordance with the hypothesis (ii). Axis $x_1 > 0$ is constituted by four orbits:

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{0},]0, a_1/a_{11} , [, { a_1/a_{11} },] a_1/a_{11} , + ∞ [,

since if $x_2 = 0$, the first equation of the system (2) is reduced to equation in (i), since if $x_2 = 0$, the first equation of the system (2) is reduced to equation in (i), whose orbits are just these sets; on the other hand axis $x₂$ is constituted by orbits:

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{0},]0, a_1/a_{11} , $\{a_1/a_{11}\}$, $\{a_1/a_{11}\}$, $\{a_1/a_{11}\}$, $\pm \infty\}$,
since if $x_2 = 0$, the first equation of the system (2) is reduced to *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
{0}, $]0, a_1/a_{11}$, $[4a_1/a_{11}$ }, $]a_1/a_{11}$, $+ \infty$ [,
since if $x_2 = 0$, the first equation of the system (2) is reduced to equation in (i), who equilibrium points, which have meaning for the ecosystem, in our case points E_1 ed E_2 . To study their stability, let's refer to the linear approximation of system (2) in a point close to the equilibrium point considered. The Jacobian matrix of matrix of the is given by on the other hand axis x_2 is constituted by orbits:

no the other hand axis x_2 is constituted by orbits:

n orbit has a point in the first quadrant, it is all included there, so that p

n not be spirals, or centers. these sets; on the other hand axis x_2 is constituted by α , $]0, +\infty[$.

efore, if an orbit has a point in the first quadrant,
 E_1, E_2 can not be spirals, or centers. Let's now m

librium points, which have meanin which has a point in the first quadrant, it is all included there, so that point
of the spirals, or centers. Let's now move to analyze the nature of isola,
which have meaning for the ecosystem, in our case points E_1 ed *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 $[0, a_i/a_{1i}, [a_i/a_{1i}], a_i/a_{1i}, +c_1],$
 $[1, x_i = 0$, the first equation of the system (2) is reduced to equation in (i), whose orbits are

sees sets; on *British Journal of Mathematics & Computer Science 4(8), 1048-1073,*
 $\{0\}$, $[0, a_i/a_{i1}, 1, {a_i/a_{i1}}]$, $[a_i/a_{i1}]$, $[a_i/a_{i$ *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 (a_{11}) ,] a_1/a_{11} , + ∞ [*x*],
 xist equation of the system (2) is reduced to equation in (i), whose orbits are

other hand axis x_2 is co **British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014**
 t_{11} , [, { a_1/a_{11} },] a_1/a_{13} , + ∞ [,

0), the first equation of the system (2) is reduced to equation in (i), whose orbits are

s; on t British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014
 $\sqrt{a_1}$, $\sqrt{a_1}$, $\sqrt{a_1}$, $\sqrt{a_1}$, $\sqrt{a_2}$,
 $= 0$, the first equation of the system (2) is reduced to equation in (i), whose orbits are
 {0},]0, *a*₁/*a*₁₁, [, {*a*₁/*a*₁₁},]*a*₁/*a*₁₁, +*z*[, since if $x_2 = 0$, the first equation of the system (2) is reduced to equation in (i), whose or ust these sets; on the other hand axis x_2 is constit a_1/a_{11} , $\lfloor a_1/a_{11} \rfloor$, $\lfloor a_1/a_{11} \rfloor$, $\lfloor a_2/a_{11} \rfloor$, $\lfloor a_3/a_{11} \rfloor$, $\lfloor a_4/a_{11} \rfloor$, $\lfloor a_5/a_{11} \rfloor$, $\lfloor a_6/a_{11} \rfloor$, $\lfloor a_7/a_{11} \rfloor$, $\lfloor a_8/a_{12} \rfloor$ is constituted by orbits.
 $\lceil a_7/a_{11} \rceil$.
 $\$ since if $x_2 = 0$, the first equation of the system (2) is reduced to equation in (i), whose orbits are
just these sets; on the other hand axis x_2 is constituted by orbits:
 $\{0\}$, $]0, +\infty[$.
Therefore, if an orbit h points, which have meaning for the ecosystem, in our case points E_1 ed E_2 . To study
 x, let's refer to the linear approximation of system (2) in a point close to the

point considered. The Jacobian matrix of matri re, if an orbit has a point in the first quadrant, it is all included there, so that points
 E_2 can not be spirals, or centers. Let's now move to analyze the nature of isolated

tim points, which have meaning for the c

$$
J(x_1, x_2) = \begin{bmatrix} a_1 - 2a_{11}x_1 - a_{12}x_2 & -a_{12}x_1 \\ -a_{21}x_2 & a_2 - a_{21}x_1 \end{bmatrix}
$$

As for point E_0 we have:

$$
J(E_0) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}
$$

linear approximation and, according to the theorem of the stability of solution of differential equation system, it is also stable for system (2) ⁰
 a_2

ch has eigenvalues $\lambda_1 = a_1, \lambda_2 = a_2$. The origin, therefore

imation and, according to the theorem of the stability c

com, it is also stable for system (2)

lculated in E_1 becomes:
 a_1
 $a_2 - a_{21} \frac{a$ s eigenvalues $\lambda_1 = a_1, \lambda_2 = a_2$. The origin, therefore,

on and, according to the theorem of the stability of

is also stable for system (2)

ted in E_1 becomes:
 $-a_{12} \frac{a_1}{a_{11}}$
 $a_2 - a_{21} \frac{a_1}{a_{11}}$
 $a_3 = a_1$

The matrix calculated in E_1 becomes:

$$
J(E_1) = \begin{bmatrix} -a_1 & -a_{12} \frac{a_1}{a_{11}} \\ 0 & a_2 - a_{21} \frac{a_1}{a_{11}} \end{bmatrix}
$$

their stability, let's refer to the linear approximation of system (2) in a point close
equilibrium point considered. The Jacobian matrix of matrix of the is given by
 $J(x_1, x_2) = \begin{bmatrix} a_1 - 2a_1x_1 - a_1x_2 & -a_1x_1 \\ -a_2x_2 & a$ considered. The Jacobian matrix of matrix of the is given by
 $-2a_{11}x_1 - a_{12}x_2 - a_{21}x_1$
 $- a_{21}x_2$ $a_2 - a_{21}x_1$

we have:

we have:

as eigenvalues $\lambda_1 = a_1, \lambda_2 = a_2$. The origin, therefore, is an unstable node thus, i.e. s i.e.d. to the interact approximation or system (x_i in a point considered. The Jacobian matrix of matrix of the is given by
 $= \begin{bmatrix} a_1 - 2a_1x_1 - a_1x_2 & -a_1x_1 \\ -a_2x_2 & a_2 - a_2x_1 \end{bmatrix}$

in E₀ we have:
 and has determinant $\neq 0$, which assures that the point is an isolated one, and has eigenvalues ¹ that is if $a/a \ge a$ quation system, it is also stable for system (2)

he matrix calculated in E_1 becomes:
 $\begin{pmatrix} (E_1) = \begin{bmatrix} -a_1 & -a_{12} \frac{a_1}{a_{11}} \\ 0 & a_2 - a_{21} \frac{a_1}{a_{11}} \end{bmatrix} \end{pmatrix}$

o $a_2 - a_{21} \frac{a_1}{a_{11}}$

and has determinant $\$ a_{11} , a_{21} , a_{11} , a_{22} int E_0 we have:
 $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$

which has eigenvalues $\lambda_1 = a_1, \lambda_2 = a_2$. The origin, therefore, is an unstable node for

proximation and, according to the theorem of the stability of solution of diffe point E_0 we have:
 $a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$
 a_2 which has eigenvalues $\lambda_1 = a_1, \lambda_2 = a_2$. The origin, therefore, is an unstable node for

a a which has eigenvalues $\lambda_1 = a_1, \lambda_2 = a_2$. The origin, therefore, i *d* $(F_1, x_2) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$
 As for point E_0 we have:
 $J(F_0) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$
 a matrix which has eigenvalues $\lambda_i = a_1, \lambda_2 = a_3$. The origin, therefore, is an unstable node for

innear app as said <0) it is a stable node.

The matrix calculated in E_2 becomes:

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$$
J(E_2) = \begin{bmatrix} a_1 - 2a_{11} \frac{H_1}{D} - a_{12} \frac{H_2}{D} & -a_{12} \frac{H_1}{D} \\ -a_{21} \frac{H_2}{D} & a_2 - a_{21} \frac{H_1}{D} \end{bmatrix} = -\frac{1}{D} \begin{bmatrix} a_{11}H_1 & a_{12}H_1 \\ a_{21}H_2 & 0 \end{bmatrix}
$$
\nit has also determinant ≠ 0 (*E*₂ isolated point) and has characteristic equation:

\n
$$
\lambda^2 - (a_{11}H_1)\lambda + (-a_{12}a_{21}H_1H_2) = 0.
$$
\nBeing *H*₂ < 0, it will have Δ > 0 and, therefore, *λ*_{1,2} real eigenvalues:

\n
$$
\lambda_{1,2} = \frac{1}{2} \left(a_{11}H_1 \pm \sqrt{(a_{11}H_1)^2 - 4DH_1H_2} \right),
$$
\nresulting one positive and one negative.

\nFollowing the above mentioned theorem on stability, *E*₂ is a saddle point, an unstable equilibrium of orbits to that of the original system, meaning that they are not one, with identical configuration of orbits to that the equilibrium point is the same. We are now able to draw the

it has also determinant $\neq 0$ (E_2 isolated point) and has characteristic equation:

$$
\lambda^2 - (a_{11}H_1)\lambda + (-a_{12}a_{21}H_1H_2) = 0.
$$

$$
\lambda_{1,2} = \frac{1}{2} \Big(a_{11} H_1 \pm \sqrt{\big(a_{11} H_1\big)^2 - 4DH_1 H_2} \Big),
$$

resulting one positive and one negative.

Following the above mentioned theorem on stability, E_2 is a saddle point, an unstable equilibrium one, with identical configuration of orbits to that of the original system, meaning that they are not geometrically identical, but that the equilibrium point is the same. We are now able to draw the frame of phases for the system in the case Fig. 5. In this case, according to the fact that the initial point is above or below the parting lines of the saddle, labeled the system or will result overwhelming, respectively, in the interaction.

Fig. 5. Configuration of orbits.

It is obvious that all the above results can be applied in the study of multicriteria optimization problems with the objective maps taking values in Hausdorff locally convex spaces ordered by

л

supernormal cones, in particular, in any normed linear (Banach) space ordered by a well based convex cone, usually in any Euclidian space ordered by customary positive pointed and cone. After all, any ecosystem is multidimensional al least with respect to the knowledge and from the mathematical point of view we appreciate that it can be tackled as a generalized dynamical system for which one looks up the (possibly approximate) equilibrium points in the following manner. Indeed, if we denote by *A* the set of all possible points $(x_1(t), x_2(t),..., x_n(t))$ for the system (1), we consider $K = R^n_+$, and the generalized dynamical system Γ_0 given by $\Gamma_0(a) = A \cap (a)$ uter Science 4(8), 1048-1073, 2014
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ary positive pointed and cone.
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 whenever $\mathcal{E} \in R_+^n \setminus \{0\}$ and $a \in A$ or, more $\Gamma_r(a) = A \cap (a - T - K)$ respectivel *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

a cones, in particular, in any normed linear (Banach) space ordered by a well based
 α , usually in any Euclidian space ordered by eustomary pos $\mathcal{O}_T(a) = A \cap [a - T - \text{int}(K)]$ with $\phi \neq T \subseteq K \setminus \{0_K\}$. **Example 10** *Bertitsh Journal of Mathematics* $\& Compute Science\ A(\delta), 1048{\text -}1073, 2014$
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concurve cone, usually in any Euclidian space o **Example 11 But the** *b Mathematics* $\& Compute$ **Science** $4(8)$, $1048-1073$, 2014 **supernormal** cones, in particular, in any formed linear (Banach) space ordered by a well based convex cone, usually in any Euclidia ven by $\Gamma_0(a) = A \cap (a - K)$,
 Γ (1). If we wish to consider

or $\Gamma_s(a) = A \cap (a - \varepsilon - K)$
 $\cap (a - T - K)$ respectively

the important difference that,
 $\Gamma_k(A) = \bigcap_{\varepsilon \in K\setminus\{0\}} [\varepsilon - MIN_k(A)]$ with

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 $y \text{ }\Gamma_0(a) = A \cap (a - K),$

If we wish to consider
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In the second last approximate special case we remark immediately the important difference that, if we denote by $\varepsilon - MIN_K(A) = \{a_0 \in A : A \cap (a_0 - \varepsilon - K) = \phi\}$, then $MIN_K(A) = \bigcap_{n=0}^{\infty} \left[\varepsilon - MIN_K(A)\right]$ with

the similar corresponding equality for \mathcal{E} - maximum points which also may represent a sort of equilibrium points under appropriate conditions submitted also to stability.

It is clear that if one denotes by $\alpha_i = \alpha_i(t)$ the weight attached on x_i $(i = 1, n)$ in (1), then any *strong minimum* for the function $f = f(t) = \sum_{i=1}^{n} \alpha_i \cdot x_i$ is in $MIN_K(A)$ whenever at least or $f = f(t) = \sum \alpha_i \cdot x_i$ is in $MIN_K(A)$ whenever at least on 1 $\alpha_i \cdot x_i$ is in $MIN_K(A)$ whenever at least one

 α _i is not zero. This performs another modality to approach the equilibrium of any (multidimensional) ecosystem, that is, using the scalarization methods. Concerning the decision – making multidimensional ecosystems approached by the splines in H-locally convex spaces, let $(X, P = \{p_{\alpha} : \alpha \in I\})$ be a H–locally convex space, that is, a separated locally convex space with the seminorms satisfying the next usual parallelogram law:

$$
p_{\alpha}^{2}(x+y) + p_{\alpha}^{2}(x-y) = 2[p_{\alpha}^{2}(x) + p_{\alpha}^{2}(y)] \ \forall x, y \in X, p_{\alpha} \in P
$$

Let us consider *M* be a closed linear subspace of *X* for which there exists a H-locally convex space $(Y, Q = \{q_\alpha : \alpha \in I\})$ and a linear (continuous) operator $U : X \to Y$ such that $M = \{x \in X : (x, y)_{\alpha} = (Ux, Uy)_{\alpha}, \forall y \in X, \alpha \in I\}$ where $(\cdot, \cdot)_{\alpha}$ $(\alpha \in I)$ denotes the scalar semiproduct which generates the seminorm $p_{\alpha} \in P$ and $\langle \cdot, \cdot \rangle_{\alpha}$ is the scalar semiproduct generating the seminorm $q_a \in \mathcal{Q}, \alpha \in I$. The linear subspace of spline functions with respect to *U* defined for the first time by us as the U-orthogonal of *M* is $M^{\perp} = \{x \in X : \langle Ux, U\zeta \rangle_{\alpha} = 0, \forall \zeta \in M, \alpha \in I\}$ Clearly, M^{\perp} is the orthogonal of *M* in the H-locally convex sense, that is,

$$
M^{\perp} = \{x \in X : \langle x, y \rangle_{\alpha} = 0, \forall y \in M\}
$$

The main result concerning the immediate important connections between the best approximation and the vectorial optimization is the following

Theorem 1.

(i) if $K = R_+^I$, then for each $s \in M^\perp$, every $\sigma \in M^\perp$ is the only solution of the *next vectorial optimization problem:* $MIN_{K} \left\{ \left\{ \left(q_{\alpha} (U(\eta - s)) \right): \eta \in M \oplus M^{\perp} \text{ and } \eta \cdot \sigma \in M \right\} \right\}$

(ii) for every
$$
x \in M \oplus M^{\perp}
$$
 its spline function (projection onto M^{\perp}) s_x is the
only solution for each of the following vectorial optimization problems:

$$
MIN_K \left(\left\{ (q_{\alpha}(U(\eta - x))) : \eta \in M^{\perp} \right\} \right),
$$

$$
MIN_K \left(\left\{ (p_{\alpha}(x - y)) : y \in M^{\perp} \right\} \right),
$$

$$
MIN_K \left(\left\{ (q_{\alpha}(U_y)) : y - x \in M \right\} \right).
$$

Example 1.

Let
\n
$$
X = \{ f \in C^{m-1}(R) : f^{(m-1)} \text{ is locally absolutely continuous and } f^{(m)} \in L^2_{loc}(R) \}, \ m \ge 1
$$

endowed with the H-locally convex topology generated by the next scalar semiproducts:

$$
(x,y)_k = \sum_{h=0}^{m-1} \left[x^{(h)}(k) \cdot y^{(h)}(k) + x^{(h)}(-k) \cdot y^{(h)}(-k) \right] + \int_{-k}^{k} x^{(m)}(t) \cdot y^{(m)}(t) dt, k = 0, 1, 2, ...
$$

and $Y = L^2_{loc}(R)$ equipped with the H-locally convex topology induced by the scalar semiproducts

$$
\langle x, y \rangle_k = \int_{-k}^k x(t) \cdot y(t) dt, \ k = 0, 1, 2, \dots
$$

If $U: X \rightarrow Y$ is the derivation operator of order *m*, then

$$
M = \left\{ x \in X : x^{(h)}(\nu) = 0, \forall h = \overline{0, m - 1}, \nu \in Z \right\} \text{ and}
$$

$$
M^{\perp} = \left\{ s \in X : \int_{-k}^{k} s^{(m)}(t) x^{(m)}(t) dt = 0, \forall x \in M, k = 0, 1, 2, ... \right\} =
$$

 $S = \{s \in X : s_{(v,v+1)} \text{ is a polynomial function of degree } 2m - 1 \text{ at most, } \forall v \in Z\}$ Any spline function *S* such as this is defined by

$$
S(x) = p(x) + \sum_{h=0}^{m-1} c_1^{(h)}(x-1)_+^{2m-1} + \sum_{h=0}^{2m-1} c_2^{(h)}(x-2)_+^{2m-1} + K + \sum_{h=0}^{m-1} c_0^{(h)}(-x)_+^{2m-1} + K
$$

where $u_+ = \frac{|v_+ - v_+|}{2}$, $\forall u \in R$, p is a polynomial function of degree 2*m*-1 at most perfect $+u$ $\psi_+ = \frac{|\mathfrak{m}| + \mathfrak{m}}{2}$, $\forall u \in R$, p is a polynomial function c u + u $u_{+} = \frac{u_{+} - u_{-}}{2}$, $\forall u \in R$, p is a polynomial function of degree 2*m*-1 at most perfectly determined together with the coefficients $c_v^h(h = \overline{0, m - 1}, \forall v \in Z)$ by the interpolation conditions.

Therefore, for every function $f \in X$ there exists an unique function denoted by $S_f \in M^{\perp}$ such that $S_f^{(h)}(\nu) = f^{(h)}(\nu)$, $\forall h = 0, m-1, \nu \in \mathbb{Z}$. Hence M and M^{\perp} give an orthogonal $f_f^{(h)}(\nu) = f^{(h)}(\nu)$, $\forall h = 0, m-1, \nu \in Z$. Hence *M* and M^{\perp} give an orthogonal decomposition for the space *X*, that is $X = M \oplus M^{\perp}$.

Example 2.

Let $X = \{f \in C^{m-1}(R) : f^{(m-1)} \text{ is locally absolutely continuous and } f^{(m)} \in L^2(R)\}$ endowed with the H-locally convex topology induced by the scalar semiproducts $(x, y)_v = x(v)y(v) + \int_R x^{(m)}(t)y^{(m)}(t)dt$, $v \in Z$, $Y = L^2(R)$ with the topology generated by the inner product $\langle x, y \rangle_{V} = \int_{R} x(t)y(t)dt$ and $U: X \to Y$ be as usually the de $\langle x, y \rangle_{V} = \int x(t)y(t)dt$ and $U: X \to Y$ be as usually the derivation operator of order *m.*

Then

$$
M = \{x \in X : x(v) = 0, \forall v \in Z\} \text{ and}
$$

$$
M^{\perp} = \{s \in X : \int_{R} x^{(m)}(t) s^{(m)}(t) dt = 0, \forall x \in M\}
$$

As in Example 1, M^{\perp} coincides with the class of all piecewiese polynomial functions of order *2m* (degree *2m-1* at most) having their knots at the integer points of the real axis.

Consequently, for every function $f \in X$, there exists an unique spline function $S_f \in M^{\perp}$ which achieves an optimal interpolation for *f* on the set *Z*, that is, S_f satisfies the equalities $S_f(v) = f(v)$ for every $v \in Z$ and it is the unique solution for each corresponding optimization problem contained in the Theorem 1.

Thus, $M \oplus M^{\perp} = X$, that is, M and M^{\perp} give an orthogonal decomposition of the space X,

 M^{\perp} being again simultaneous and vectorial proximinal with respect to the family of seminorms generated by the above scalar semiproducts. Now it is very clear that our abstract construction for the splines and the above examples suggest another possibility for the study and analysis of the decision–making long-time problems concerning the multidimensional ecosystems whenever it is possible to approximate or to know in fact their demeanour for the criteria and the constraints point of view, because for any objective functions we exhibited the possibility to assessed its subsequent behaviour by the corresponding spline functions, under reasonable degrees of regularity. Now we present some open problems, conclusions and remarks. Upon close examination, the on top considerations lead to the following open problems and the new possibilities to tackle the multidimensional ecosystem programs from the mathematical point of view – concerning the optimality, with the inherent typical open problems:

- 1. Convergence methods on the existence of the solutions and the specific stability for the equilibrium points sets in the most general models of ecosystems.
- 2. Stability and approximate stability for the evolution of the ecosystems.
- 3. Mathematical methods governing the life of ecosystems.
- 4. The Pareto type efficiency (with its corresponding particular) cases of strong or contrivance between strong and vector optimization) can be applied for the investigation of the multidimensional decision – making ecosystem problems (see Section 2 and Section 3 of the present research paper).
- 5. The abstract method generating, in particular, the cardinal spline functions presented in Example 1 and Example 2 are very useful to find optimal interpolation solutions especially for the long-time decision-making multidimensional ecosystems programs whenever it is known their previous and/or the present behaviour and one desires to check the further behaviours under permissible and suitable degrees of smoothness for the objective functions.
- 6. The vectorial or approximate optimality versions seems to be more adequated than the strong (simultaneous) most propitious solutions, but this will be seen in the further studies because the natural context suggest rather permanent time optimal combinations between these modalities of study.
- 7. Every preliminary analysis can be very relevant and must take priority before implementing any model of multicriteria analysis or which aims at resolving conflictuality.

5.2 New Proposals for the Study of the Equilibria in the Fish Wars

This section is devoted to the study of some types of equilibrium, under the types interdependences between different species of fish and in the context of biological and dynamic interactions. Techniques from splines in H-locally convex spaces and applications and Pareto efficiency are suggested to be used for these investigations.

The main aim of this paper is to present some new possibilities to investigate the equilibria in the fish wars through the agency of our spline functions and the recent results on the efficiencythe important results on Pareto optimization in the general context of ordered Hausorff locally convex spaces.For the interactions of fish species and the related topics, let us assume that there are

British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014
 $p \in N^*$ owners and each of whom can fish $n \in N^*$ fish species. If one denotes by
 $(x_{1i}, x_{2i}, ..., x_{ni})$ the stocks of each kind of fish at the momen *British Journal of Mathematics* & *Computer Science* 4(8), 1048-1073, 2014
 $p \in N^*$ owners and each of whom can fish $n \in N^*$ fish species. If one denotes by
 $(x_{11}, x_{21},...,x_{ni})$ the stocks of each kind of fish at the mom instance [4] and its references for n=p=2) that the general biological growth is described by *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 $p \in N^*$ owners and each of whom can fish $n \in N^*$ fish species. If one denotes by
 $\langle x_{i_1}, x_{i_2},..., x_{i_n} \rangle$ the stocks of each kind of fish at th & Computer Science 4(8), 1048-1073, 2014

V^{*} fish species. If one denotes by

moment t, then it is known (see, for

iological growth is described by
 $i = \overline{1, n}$.
 $i = \overline{1, n}$. Mathematics & Computer Science 4(8), 1048-1073, 2014

fish $n \in N^*$ fish species. If one denotes by

fish at the moment *t*, then it is known (see, for

ne general biological growth is described by
 $\sum_{i}^{p} = \sum_{j=1}^{p} c_{$ *CMathematics & Computer Science 4(8), 1048-1073, 2014*

fish $n \in N^*$ fish species. If one denotes by

f fish at the moment t, then it is known (see, for

the general biological growth is described by
 $c_{ii} = \sum_{j=1}^p c_{j$ *uhematics* & *Computer Science* 4(8), 1048-1073, 2014

sh $n \in N^*$ fish species. If one denotes by

sh at the moment *t*, then it is known (see, for

general biological growth is described by
 $= \sum_{j=1}^p c_{jtt}$, $i = \overline{1$ *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 $p \in N^*$ owners and each of whom can fish $n \in N^*$ fish species. If one denotes by
 $(X_i, X_2, ..., X_n)$ the stocks of each kind of fish at the moment *t* **British Journal of Mulkenaites & Computer Science 4(8), 1048-1073, 2014**
 $p \in N^*$ owners and each of whom can fish $n \in N^*$ fish species. If one denotes by
 $(x_{i_1}, x_{i_2}, ..., x_{i_m})$ the stocks of each kind of fish at the mo British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014

of whom can fish $n \in N^*$ fish species. If one denotes by

of each kind of fish at the moment t, then it is known (see, for

for n=p=2) that the gen *Brttish Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 N^{*} owners and each of whom can fish $n \in N^*$ fish species. If one denotes by
 $x_2, ..., x_n$) the stocks of each kind of fish at the moment *t*, th *British Journal of Mathematics* & *Computer Science 4(8)*, 1048-1073, 2014

s and each of whom can fish $n \in N^+$ fish species. If one denotes by

the stocks of each kind of fish at the moment *t*, then it is known (see, *p i* **i i** *i i*

under the total catch of each species given by $c_{i} = \sum_{i=1}^{n} a_{i}$ 1 *p j*

Consequently, each period's population is characterized by

$$
x_{it+1} = f_i(x_{1t} - c_{1t}, x_{2t} - c_{2t},..., x_{nt} - c_{nt}), i = \overline{1,n}
$$

and if we suppose, in the natural background, that the utility from consuming the *n* species of fish corresponding sum of the discounted utility, that is, to solve the following problem the total catch of each species given by $c_{ii} = \sum_{j=1}^{n} c_{jii}$, $i = 1, n$.

quently, each period's population is characterized by
 $= f_i (x_{1t} - c_{1t}, x_{2t} - c_{2t}, ..., x_{ni} - c_{ni})$, $i = \overline{1, n}$

we suppose, in the natural backgroun Let the lotation of each species given by $c_{ii} - \sum_{j=1}^{n} c_{jii}$, $i = 1, n$

Lently, each period's population is characterized by
 $f_i(x_{1t} - c_{1t}, x_{2t} - c_{2t},..., x_{nt} - c_{nt})$, $i = \overline{1,n}$

we suppose, in the natural background, t *is* $u(c_{j1}, c_{j2},...,c_{jnt})$, $j = \overline{1, p}$, then every $j = \overline{1, p}$ of owners is interested to maximize the
corresponding sum of the discounted utility, that is, to solve the following problem
(P_j) $\lim_{(c_{j1}, c_{j2},...,c_{jn})} \sum_{t$

$$
(\mathbf{P}_{j}) \prod_{(c_{j1}, c_{j2}, \dots, c_{jnl})} \sum_{t=1}^{\infty} \delta_{j}^{t} u\Big(c_{j1t}, c_{j2t}, \dots, c_{jnt}\Big), 0 < \delta_{j} < 1, j = \overline{1, p}.
$$

In the next considerations, we propose new modalities for the research investigations of these instance [4] and its references for n=p=2) that the general biological growth is described by $x_{n+1} = f_i(x_n, x_2, ..., x_m)$, $i = \overline{1, n}$
under the total catch of each species given by $c_a = \sum_{j=1}^{p} c_{jn}$, $i = \overline{1, n}$.
Conseque to be analysed like multidimensional ecosystems as follows: first of all, we remark that, taking into account known values for the objective functions (and for some consecutive derivatives when these exist) defined by $\lim_{(c_{j1}, c_{j2}, \dots, c_{jn})} \sum_{t=1}^{\infty} \delta'_j u(c_{j1}, c_{j2}, \dots, c_{jnt}), 0 < \delta_j < 1, j = \overline{1, p}.$

enever $x_n \ge c_{ii} \ge 0$ and $x_{n+1} = f_i(x_{1i} - c_{1i}, x_{2i} - c_{2i}, \dots, x_{ni} - c_{ni})$ for any $i = \overline{1, n}.$

the next considerations, we propose new modali iod's population is characterized by
 $u_x - c_{2x},..., x_m - c_m$), $i = \overline{1, n}$

be natural background, that the utility from consuming the *n* species of fish
 $\sum_{i=1}^{n} \overline{L_i}$, then every $j = \overline{1, p}$ of owners is interested t *j* $\sum_{(c_{j1t}, c_{j2t}, \dots, c_{jm})}^{MAX} \sum_{t=1}^{\infty} \delta'_j u(c_{j1t}, c_{j2t}, \dots, c_{jnt})$, $0 < \delta_j < 1$, $j = \overline{1, p}$.

henever $x_{il} \ge c_{il} \ge 0$ and $x_{il+1} = f_i(x_{1t} - c_{1t}, x_{2t} - c_{2t}, \dots, x_{nt} - c_{nt})$ for any $i = \overline{1, n}$

the next considerations, time the total case of each species given by $c_n = \sum_{i=1}^{n} c_{ni}t = 1/n$.

Consequently, each period's population is characterized by
 $x_{n+1} = f_i\left(x_n - c_{i_1}, x_{2i} - c_{2k}, ..., x_m - c_m\right)$, $i = \overline{1,n}$

and if we suppose, in the natura and if we suppose, in the natural background, that the utility from consuming the *n* species of fish

is $u(e_{j_1j_1}, e_{j_2j_2},...,e_{j_{j_{j}}}, f) = \overline{1, p}$, then every $j = \overline{1, p}$ of owners is interested to maximize the

corres in the natural background, that the utility from consuming the *n* species of fish
 $c_{j\omega}$, $j = \overline{1, p}$, then every $j = \overline{1, p}$ of owners is interested to maximize the

of the discounted utility, that is, to solve the presponding sum of the discounted utility, that is, to solve the following problem
 *P*₁ $\sum_{(s_1, a, c_2, ..., a_N)} \sum_{j=1}^{\infty} \delta_j^s u\left(c_{j11}, c_{j22},...,c_{jm}\right), 0 < \delta_j < 1, j = \overline{1, p}$.

henever $x_a \ge c_n \ge 0$ and $x_{a+1} = f_1\left(x_{n_1} - c_{$

$$
\varphi_j(t) = \sum_{s=1}^t \delta_j^s u(c_{j1s}, c_{j2s}, ..., c_{jns}), j = \overline{1, p}, t \in [1, +\infty)
$$

during a previous time period it is possible to use our splines in order to approximate and optimal interpolate them. It is clear that this procedure depends on the degrees of regularity for any strong individual or the strong global maximization, respectively to solve the problem

$$
(P)=\bigcap_{j=1}^p(P_j).
$$

Also, it is obvious that any strong global maximization solution concerning the equilibrium is a strong individual maximization equilibrium. From the vector optimization point of view which we consider more appropriate for the "reconciliation" of the real factors the global problem can be studied from the efficiency point of view:

if we denote by *A* the set of all

British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014\nconsider more appropriate for the "reconcilation" of the real factors the global problem can be
\nstudied from the efficiency point of view:
\nif we denote by *A* the set of all\n
$$
\varphi = (\varphi_1(t), \varphi_2(t), ..., \varphi_p(t))
$$
\n
$$
\left(\sum_{t=1}^{\infty} \delta_1^t \cdot u\left(c_{11}, c_{12}, ..., c_{1m}\right), \sum_{t=1}^{\infty} \delta_2^t \cdot u\left(c_{21}, c_{22}, ..., c_{2m}\right), ..., \sum_{t=1}^{\infty} \delta_p^t \cdot u\left(c_{p1}, c_{p2}, ..., c_{pm}\right)\right)
$$
\nrespectively, we consider $K = R_+^p$ and the generalized dynamical system $\Gamma_0: A \to 2^A$ given
\nby $\Gamma_0(a) = A \cap (a + K)$, then any (approximate) equilibrium point for Γ_0 , represents an
\n(approximate) equilibrium for our problem $\left(P_1, P_2, ..., P_p\right)$.
\nIf we wish to continue the study, then we can take $\Gamma_1(a) = A \cap [a + \text{int}(K)]$ or
\n $\Gamma_r(a) = A \cap (a + T + K)$ respectively $\Gamma_r^0 = A \cap [a - T - \text{int}(K)]$ with $\varnothing \neq T \subseteq K$. For

respectively, we consider $K = R_+^p$ and the generalized dynamical system $\Gamma_0 : A \to 2^A$ given

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consider more appropriate for the "reconciliation" of the real factors the global problem can be

studied from the efficiency point of view:
 φ **Both to continue the study,** then \mathcal{H}_{M} and \mathcal{H}_{M} and \mathcal{H}_{M} and \mathcal{H}_{M} are \mathcal{H}_{M} and \mathcal{H}_{M} and \mathcal{H}_{M} are \mathcal{H}_{M} and \mathcal{H}_{M} are denote by \mathcal{H} the studied from the efficiency *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*
 Existing the oriental of the "reconditionion" of the real factors the global problem can be

strated from the efficiency point of view:
 $\varphi = (\var$ respectively $\Gamma^0_T = A \cap [a - T - \text{int}(K)]$ with $\varnothing \neq T \subseteq K$. For *rnal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

meiliation" of the real factors the global problem can be
 $\iota(c_{21t}, c_{22t},...,c_{2m})$,..., $\sum_{i=1}^{\infty} \delta'_p \cdot u(c_{p1t}, c_{p2t},...,c_{pm})$

d the generalized dynamical s *^T* and *^K* \ 0 *^K* taking *eff A K a A A a K* 1 0 0 : one obtains $eff(A,-K) = \bigcap \left[\varepsilon - eff(A,-K)\right]$, that is, another posibility to approxim $H = R_+^p$ and the generalized dynamical system $\Gamma_0 : A$ -
then any (approximate) equilibrium point for Γ_0 , re
our problem $(P_1, P_2, ..., P_p)$.
e study, then we can take $\Gamma_1(a) = A \cap [a + \text{in}$
ectively $\Gamma_1^0 = A \cap [a - T - \text{int}(K)]$ *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

re appropriate for the "reconciliation" of the real factors the global problem can be

the efficiency point of view:

by *A* the set of all
 P , ε ε *British Journal of Mathematics & Computer Science 4(8), 1048-1073, 2014*

opriate for the "reconciliation" of the real factors the global problem can be

dictincy point of view:
 $\varphi_p(t)$
 $\sum_{i=1}^n \delta'_2 \cdot u(C_{1i}, C_{22i}, ..., C_{$ vectorial equilibria points.

6 Conclusion

This research work represents an original scientific contribution for the study of general equilibria, It can be developed with appropriate numerical methods for mathematical programming concerning efficiency, optimization and conversely.

Competing Interests

Author has declared that no competing interests exist.

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