British Journal of Mathematics & Computer Science
6(3): 187-203, 2015, Article no.BJMCS.2015.072
ISSN: 2231-0851

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Fractional Variational Iteration Method for Fractional Fornberg-Whitham Equation and Comparison with the Undetermined Coefficient Method

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Article Information

DOI: 10.9734/BJMCS/2015/15392
Editor(s):
(1) Metin Başarir, Department of Mathematics, Sakarya University, Turkey.
Reviewers:
(1) Roshdi Khalil, Department of Mathematics, University of Jordan, Jordan.
(2) Anonymous, Pakistan.
(3) Anonymous, Turkey.
(4) Jianfei Huang, College of Mathematics, Qingdao University, China.
Complete Peer review History: http://www.sciencedomain.org/review-history.php?iid=733&id=6&aid=7607

Original Research Article

Received: 24 November 2014
Accepted: 18 December 2014
Published: 03 January 2015

Abstract

The paper presents two methods for solving the fractional Fornberg-Whitham (FFW) equation. Based on the peaked solutions of FW equation, suppose the solution's variable-separated form, and the FFW equation is transformed into a constant fractional differential equation (FDE). To solve the transformed equation, first, the fractional variational iteration method (FVIM) is used. Secondly, the undetermined coefficient method is used to expand the solution of the constant FDE. The expansion is based on the Generalized Taylor formula. Also the solutions are yielded for FFW. It should be pointed out that two cases of the order of fractional derivative between 1 and 2 and that between 0 and 1 are discussed respectively for the transformed FDE. Last, we give two numerical examples by using the two presented methods. The results show that the results agree well by both two proposed methods, and the two methods are high efficient in solving FFW.

Keywords: Differential transform method; Generalized Taylor formula; Caputo fractional derivative; Fractional differential equations; fractional Fornberg-Whitham equation.

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1 Introduction

During the past three decades, great interest has been attracted in many phenomena and problems which can be dealt with by using the fractional derivative in engineering, physics, biology, fluid mechanics and other sciences. From the aspect of physical and engineering problems, there are many applications such as control theory of dynamical systems, creeping and relaxation for viscoelastic materials, diffusion and reaction processes, electrical networks, optics and signal processing [1-3].

Considerable effort has been devoted to find robust and stable numerical and analytical methods for solving fractional differential equations. Some numerical methods [4–8] and analytical methods [9–10] have been developed for fractional differential equations (FDEs).

Other newly developed methods in dealing with the partial order differential equations of fractional order such as modified simple equation method, the first integral method, (G/G)-expansion method and *et al* can be found in [11-14].

The variational iteration method (VIM) is a generally used approach for finding the approximate solution of linear and nonlinear problems. VIM was proposed by He [15-16] who was the first to apply the VIM to fractional differential equations in [16].

Many scholars have applied VIM or modified version to solve linear or nonlinear differential equations [17-20]. For a relatively comprehensive survey on the concepts, theory and applications of the method, readers are referred to review articles [21-22].

The late development of the VIM in the fractional calculus is q-difference equation and fractional one [23-24].

In this paper, we consider the following time-fractional Fornberg–Whitham equation (FFW equation)

$$ {}_0^C D_t^\alpha u - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \quad t>0,\ \alpha>0 \tag{1}$$

with initial condition as

$$u(x,0) = Ae^{\frac{1}{2}x} \tag{2}$$

where α is the parameter describing the order of the fractional Fornberg–Whitham equation, and A is an arbitrary constant. The fractional derivatives are understood in the Caputo sense.

It is obvious that the response expression should contain the order parameter of the fractional derivatives α . In the case of $\alpha=1$ the fractional Fornberg–Whitham equation reduces to the classical nonlinear Fornberg–Whitham equation. It's known that Fornberg and Whitham obtained a peaked solution for Eqs. (1) and (2) with the form:

$$u(x,t) = A\mathrm{e}^{\frac{1}{2}\left|x-\frac{4}{3}t\right|} \tag{3}$$

The nonlinear Fornberg–Whitham equations have been studied by many authors such as Saker et al. [25] and Lu [26].

Recently, Laplace transform is adopted in some famous analytical methods to simplify the solution process and improve solution's accuracy. HPTM is used by Merdan etc. in [27] and Jagdev Singh etc. in [28] to solve the FFW equation. Besides, the Laplace Adomian decomposition method (LAPM) is also used in [29,30] (Tsai and Chen, 2010; Zeng and Qin, 2012). Especially, it should be pointed that Wu and Dumitru firstly suggest a universal way to identify the multiplier by implementing Laplace transform. The way is proved to be a simple but effective approach and used in many problems [31-34].

In the paper, aiming at solving the FFW equation, we use the fractional iteration method (FVIM) with the Lagrange multiplier determined by the way in [32-35]. Besides, we use undetermined coefficient method for solving the nonlinear FFW equation based on Generalized Taylor formula. The results agree surprising well by using the two proposed methods.

Though many authors have studied the fractional F-W equation and obtained many useful and valuable conclusions [25,35]. However, they didn't discuss the case of the fractional order larger than 1. Further, we also point out an error in the existed literatures and we give a full conclusion with two cases of the fractional order's range.

For the convergence of the Fractional variational iteration method, the convergence criterion can be obtained as shown in the article [36,37], and an accelerated convergence will be obtained by the optimal variational iteration method [38].

The paper is organized as follows. In Section 3, we introduce the main idea of the variational iteration method. Approximate results for FW equation are presented. In Section 4, we apply the undetermined coefficient method for solving the nonlinear FW equation. In the last section, Discussions of cases about different parameter α (order of the FW equation) are given in order to assess the two methods: FVIM and the undetermined coefficient method. The number of items in the series solution truncated to keep the convergence is also analyzed.

2 Preliminaries

Definition 1.1. The Caputo derivative is given as

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) \mathrm{d}\tau, t > a, n-1 < \alpha < n \in Z^+ \tag{4} $$

Definition 1.2. The Riemann–Liouville (R–L) integration of $f(t)$ is defined as

$$ {}_a J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \alpha > 0 \tag{5} $$

Definition 1.3. Laplace transform of the term ${}_0^C D_t^{\alpha} u$ is given as

$$ L[{}_0^C D_t^{\alpha} u] = s^{\alpha} \overline{u}(s) - \sum u^{(k)}(0^+) s^{\alpha-1-k}, n-1 < \alpha \le n \tag{6} $$

where L is Laplace transform and $\overline{u}(s) = L[u(t)]$.

Assuming $\overline{h}(s) = L[h(t)]$ and $\overline{g}(s) = L[g(t)]$, the convolution theorem is

$$h(t)^* g(t) = \int_0^t h(t-\tau) g(\tau) \mathrm{d}\tau \tag{7}$$

and

$$\bar{h}(s)\bar{g}(s) = L[h(t)^* g(t)] \tag{8}$$

2 Solutions by FVIM Using the New Lagrange Multiplier

First we briefly introduce the Lagrange multiplier identified by Laplace Transform and the corresponding integration formula. These can also be found in [32] and other works of Wu and Dumitru.

We consider a general FDE

$${}_0^C D_t^\alpha u + R[u] + N[u] = g(t) \tag{9}$$

where $R[u]$ is a linear term and $N[u]$ is a nonlinear one.

Take Laplace transform on the correction functional Eq. established via the R-L integration, Wu and his cooperators [32] give a new way to identify the Lagrange multiplier and improves the variational iteration formula (2) as

$$u_{n+1} = u_n + {}_0 J_t^\alpha \lambda(t,\tau)\left[{}_0^C D_t^\alpha u_n + R[u_n] + N[u_n] - g(\tau)\right], \quad t>0, \alpha>0 \tag{10}$$

where $R[u_n], N[u_n]$ are restricted variation, i.e. $\delta R[u_n]=0, \delta N[u_n]=0$. The Lagrange multiplier is determined as $\lambda(t,\tau) = -1$.

The above iteration formula (10) is also valid for differential equations when α is an arbitrary positive integer.

Eq. (10) is expressed through Riemann–Liouville (R–L) integration. In fact it can be applied in the classical Riemann integration

$$u_{n+1} = u_n(s) + \int_0^t \lambda(t,\tau)\left[{}_0^C D_t^\alpha u_n + R[u_n] + N[u_n] - g(\tau)\right] \mathrm{d}\tau, \quad t>0, \alpha>0 \tag{11}$$

where $R[u_n], N[u_n]$ are restricted variation. Then the Lagrange multiplier can be identified as

$$\lambda(t,\tau) = \frac{(-1)^\alpha (\tau - t)^{\alpha-1}}{\Gamma(\alpha)} \tag{12}$$

It should be pointed out that the FVIM is introduced for fractional differential equation in this section. However, The form of Eq.(11) can be further extended to the case for partial fractional differential equation as

$$u_{n+1}(x,t) = u_n(x,t) + {}_0 J_t^\alpha \lambda(t,\tau)\left[{}_0^C D_t^\alpha u_n(x,\tau) + R[u_n(x,\tau)] + N[u_n(x,\tau)] - g(x,\tau)\right] \tag{13}$$

To simplify the FFW equation, according to the initial condition Eq.(2),we suppose

$$u(x,t) = Ae^{\frac{x}{2}}v(t) \tag{14}$$

Substituting Eq. (14) into Eq. (1), the obtained equation can be simplified as

$${}_0^C D_t^\alpha v - \frac{1}{4}v_t + \frac{1}{2}v = 0 \tag{15}$$

According to Eq. (2) and (13), the initial condition of Eq. (14) is obtained as

$$v(0) = 1 \tag{16}$$

Next we use the FVIM to solve the differential equation (15). To determine the higher order of the derivative, the value of α should be compared with 1. Two cases of $0 < \alpha < 1$ and $\alpha \geq 1$ are discussed as follows.

2.1 Solution for $0 < \alpha < 1$ by VIM

Eq.(15) can be written as

$$v_t - 4{}_0^C D_t^\alpha v - 2v = 0 \tag{17}$$

Eq. (17) is a classical one-order differential equation with fractional derivative. In the classical iteration method, the Lagrange multiplier, therefore, can be identified as λ = −1. The following variational iteration formula can be obtained

$$v_{n+1}(t) = v_n(t) - \int_0^t \left[v_{n\tau}(\tau) - 4{}_0^C D_\tau^\alpha v_n(\tau) - 2v(\tau) \right] \mathrm{d}\tau \tag{18}$$

After several iteration based on formula (18) and initial value (16), we can obtain the following approximate expressions as

$$\begin{aligned} &v_0(t) = 1 \\ &v_1(t) = 1 + 2t \\ &v_2(t) = 1 + 2t + 8\frac{1}{\Gamma(3-\alpha)}t^{2-\alpha} + 2t^2; \end{aligned} \tag{19}$$

$$v_3(t) = 1 + 2t + 8\frac{1}{\Gamma(3-\alpha)}t^{2-\alpha} + 2t^2 + 32\frac{1}{\Gamma(4-2\alpha)}t^{3-2\alpha} + 32\frac{1}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{4}{3}t^3$$

……

With the help of Mathematica software, one can obtain the rest of the approximations.

From Eq. (19), it's found that the items have the form as $t^{i-j\alpha},(i\in Z, j=0,1,\cdots,i-1)$ in the i-th iteration solution $v_i(t)$. The solutions of the FFW equation can be approximated as $u_i(x,t)=Ae^{\frac{x}{2}}v_i(t)$, $(i=0,1,2,\cdots,n-1)$.

2.2 Solution for $1\le\alpha<2$ by FVIM

When $\alpha\ge1$, Eq. (15) is a FDE with its fractional derivative order not less than 1. By using Eq. (11) and (12)，the following variational iteration formula can be obtained

$$v_{n+1}(t)=v_n(t)-\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[{}_0^CD_\tau^\alpha v_n(\tau)-\frac{1}{4}v_{n\tau}(\tau)+\frac{1}{2}v_n(\tau)\right]\mathrm{d}\tau \tag{20}$$

We start with an initial approximation $v_0(t)=v_0(0)=1$ given in Eq. (16), by the above iteration formula (20), we can obtain the other components as

$$\begin{aligned}
&v_0(t)=1\\
&v_1(t)=1-\frac{1}{2\Gamma(1+\alpha)}t^\alpha\\
&v_2(t)=1-\frac{1}{2\Gamma(1+\alpha)}t^\alpha-\frac{1}{8\Gamma(2\alpha)}t^{2\alpha-1}+\frac{1}{4\Gamma(1+2\alpha)}t^{2\alpha}
\end{aligned} \tag{21}$$

$$v_3(t)=1-\frac{1}{2\Gamma(1+\alpha)}t^\alpha-\frac{1}{8\Gamma(2\alpha)}t^{2\alpha-1}+\frac{1}{4\Gamma(1+2\alpha)}t^{2\alpha}-\frac{1}{32\Gamma(3\alpha-1)}t^{3\alpha-2}+\frac{1}{8\Gamma(3\alpha)}t^{3\alpha-1}-\frac{1}{8\Gamma(3\alpha)}t^{3\alpha}$$

……

From Eq.(21), it's found that items have the form as $t^{i\alpha-j},(i\in Z, j=0,1,\cdots,i-1)$ in the i-th iteration solution $v_i(t)$ .The solutions of the FFW equation can also be approximated as $u_i(x,t)=Ae^{\frac{x}{2}}v_i(t)$.

Particularly, when $\alpha=1$， $A=\frac{4}{3}$, the obtained $u_2(x,t)$ and $u_3(x,t)$ based on Eq.(21) are

$$u_2(x,t)=\frac{4}{3}e^{\frac{x}{2}}\left(1-\frac{5}{8}t+\frac{1}{8}t^2\right)$$

$$u_3(x,t)=\frac{4}{3}e^{\frac{x}{2}}\left(1-\frac{21}{32}t+\frac{3}{16}t^2-\frac{1}{48}t^3\right)$$

which are the same with those in [26]。

3 Solutions by Undetermined Coefficient Method Based on Generalized Taylor formula

To further discuss the FFW equation, the undetermined coefficient method is used to find the solution.

For the both two cases of $1 \le \alpha < 2$ and $0 < \alpha < 1$, Eq. (15) can be treated as the fractional constant differential equation of v about t. Using the general Taylor formula, we suppose

$$v(t) = \sum_{i=0}^{\infty} a_i \frac{t^{i\theta}}{\Gamma(1+i\theta)} \tag{22}$$

where θ is the maximum mutual of alpha and 1, and $\alpha = m\theta, n\theta = 1, m, n \in Z^+$。Here suppose $m < n$, then Eq.(15) can be written as

$${}_0^C D_t^{m\theta} v - \frac{1}{4}\left({}_0^C D_t^{n\theta} v\right) + \frac{1}{2} v = 0 \tag{23}$$

Substituting Eq. (22) into Eq. (23), it leads to

$$\begin{aligned} &\sum_{i=1}^{m-1}\left(a_{m-i} - \frac{1}{4}a_{n-i}\right)\frac{t^{-i\theta}}{\Gamma(1-i\theta)} + \sum_{i=m}^{n-1}\left(-\frac{1}{4}a_{n-i}\right)\frac{t^{-i\theta}}{\Gamma(1-i\theta)} \\ &+\sum_{i=0}^{\infty}\left(a_{m+i} - \frac{1}{4}a_{n+i} + \frac{1}{2}a_i\right)\frac{t^{i\theta}}{\Gamma(1+i\theta)} = 0 \end{aligned} \tag{24}$$

Let the coefficients of the power of t with negative exponent be zeros

$$a_{i-n+m} - \frac{1}{4}a_i = 0 \quad (i = n-m+1, n-m+2, \cdots, n-1) \tag{25}$$

$$a_i = 0 \quad (i = 1, 2, \cdots, n-m) \tag{26}$$

From Eq. (25) and (26), we can obtain

$$a_i = 0, \quad (i = 1, 2, \cdots, n-1) \tag{27}$$

If $n < m$, the corresponding results will be

$$a_i = 0 \quad (i = 1, 2, \cdots, m-1) \tag{28}$$

The uniform results for the two cases of $1 \le \alpha < 2$ and $0 < \alpha < 1$ will be

$$a_i = 0, \quad (i = 1, 2, \cdots, \max(m, n) - 1) \tag{29}$$

Let the coefficients of the power of t with non-negative exponent be zeros

$$\begin{cases} a_m - \frac{1}{4}a_n + \frac{1}{2}a_0 = 0 \\ a_{m+1} - \frac{1}{4}a_{n+1} + \frac{1}{2}a_1 = 0 \\ a_{m+2} - \frac{1}{4}a_{n+2} + \frac{1}{2}a_2 = 0 \\ \cdots \end{cases} \tag{30}$$

The general coefficient equations can be written as

$$4a_{m+i} - a_{n+i} + 2a_i = 0 \quad (i = 0,1,2,\cdots) \tag{31}$$

If $\alpha = 0.4$, then $m = 2, n = 5$ and $\theta = 0.2$. The first $p+1$ equations in Eq. (30) $(i = 0,1,2,\cdots,p)$ can be written in matrix form as follows

$$\begin{bmatrix} 0 & 4 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 2 & 0 & 4 & & & -1 & \cdots & 0 \\ & 2 & 0 & 4 & & & -1 & 0 \\ & & 2 & 0 & 4 & & \cdots & 0 \\ & & & 2 & 0 & 4 & \cdots & 0 \\ & & & & 2 & 0 & \cdots & 0 \\ & & & & & 2 & \cdots & 0 \\ & & & & & & \cdots & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ \vdots \\ a_{p+5} \end{bmatrix} = \begin{bmatrix} -2a_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \tag{32}$$

The coefficient a_0 is equal to 1 given by the initial condition, and according Eq.(29) $a_i = 0 \quad (i = 1,2,\cdots,n-1)$, then all the coefficient of $a_i \ (i = 1,2,\cdots,p+5)$ can be determined theoretically by Eq.(32), so the expansion expression of $v(t)$ can be obtained.

4 Numerical examples when $\alpha = 0.4$ and $\alpha = 1.2$

4.1 Solution for $\alpha = 0.4$ and $\alpha = 1.2$ by FVIM

When $\alpha = 0.4$, using the results in Eq. (19) by iteration Eq. (18), Fig. 1 shows the approximations of $v_n(t)$'s curve with time $t \in (0, 0.6)$ for different iteration times $n = 10, 20, 25, 28$. Then Fig. 2 shows the approximations of $v_n(t)$ changing with $t \in (0,1)$ for different iteration times $n = 20, 25, 28, 30$. Fig. 3 gives u(x,t)'s curve surface varying with t and x after 30 iteration times when $\alpha = 0.4$.

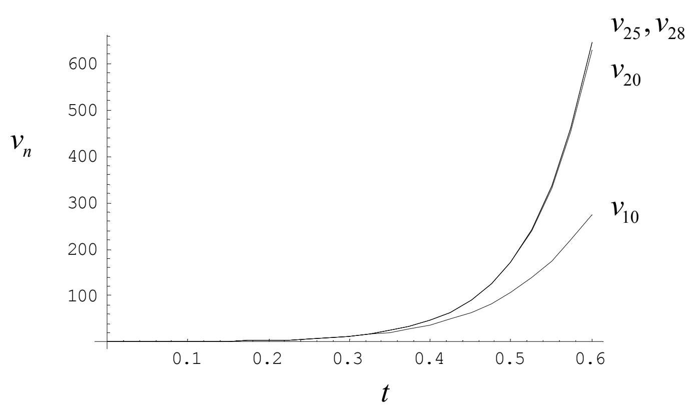

Fig. 1. v(t)'s curve by FVIM when $\alpha = 0.4$, $t \in [0, 0.6]$

When $\alpha = 1.2$, using the results in Eq. (21) by iteration Eq. (20), Fig. 4 shows the approximations of $v_n(t)$'s curve with time $t \in [0,2]$ for different iteration times $n = 1, 2, \cdots, 10$. Then Fig. 5 shows $u_{10}(x,t)$'s curve surface when $t \in [0,2]$ and $x \in [-2,2]$.

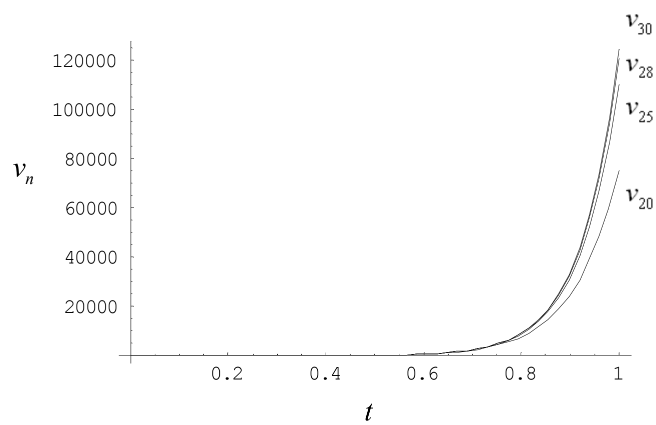

Fig. 2. v(t)'s curve by FVIM when $\alpha = 0.4$, $t \in [0,1]$

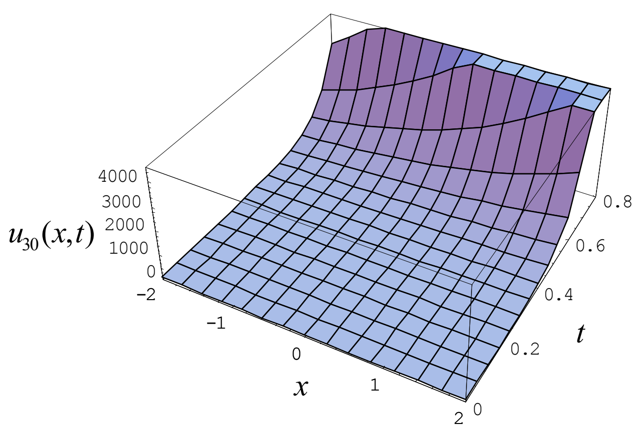

Fig. 3. u_{30}(x,t)'s curve surface varying with t and x when $\alpha = 0.4$

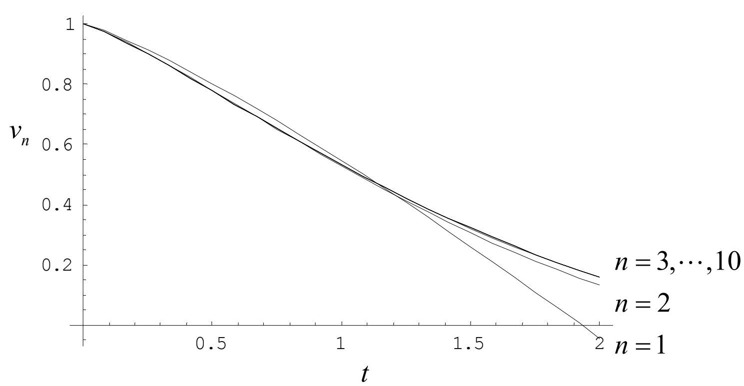

Fig. 4. v(t)'s curve by FVIM when $\alpha = 1.2$, $t \in [0,2]$

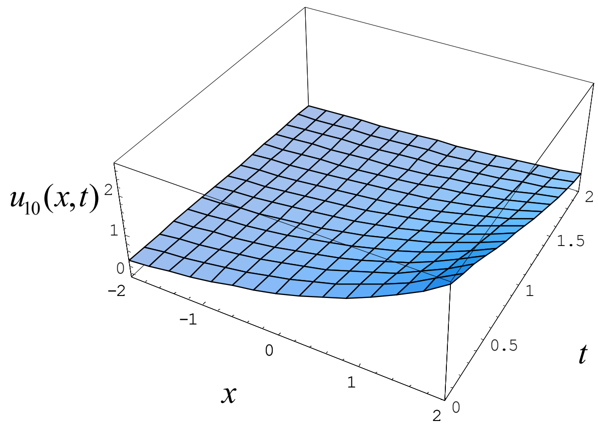

Fig. 5. u(x,t)'s curve surface varying with t and x when $\alpha = 1.2$

4.2 Solution for $\alpha = 0.4$ by the undetermined coefficient method

When $\alpha = 0.4$, then m, n, θ in Eq.(20) are $m = 2, n = 5$ and $\theta = 0.2$ respectively. The initial values of a_i are obtained in section 4 as $a_0 = 1, a_i = 0 (i = 1, 2, 3, 4)$. From Eq.(31), the iteration relationship is established for the coefficients of a_i as

$$a_{5+i} = 4a_{2+i} + 2a_i \qquad (i = 0, 1, 2, \cdots) \tag{33}$$

By using an iteration program, the values of a_i $(i = 5, 6, \cdots, 22)$ are shown as Table 1.

The expression of $v(t)$ in Eq.(20) with 10 items shown is as follows

$$v(t) = 1 + 2t + \frac{8}{\Gamma(2.6)} t^{1.6} + 2t^2 + \frac{32}{\Gamma(3.2)} t^{2.2} + \frac{32}{\Gamma(3.6)} t^{2.6} + \frac{128}{\Gamma(3.8)} t^{2.8} + \frac{4}{3} t^3 + \cdots \tag{34}$$

The approximate expression of $v(t)$ in Eq.(34) is

$$\begin{aligned} v(t) = 1 + 2t + 5.59587t^{1.6} + 2t^2 + 13.2015t^{2.2} + 8.60904t^{2.6} \\ + 27.2678t^{2.8} + 1.33333t^3 + 24.7528t^{3.2} + 50.5125t^{3.4} + \cdots \end{aligned} \tag{35}$$

When $\alpha = 0.4$, Figs. 6-8 show that $v(t)$'s curves by the undetermined coefficient method for three time intervals: $[0, 0.3]$, $[0, 0.6]$ and $[0.6, 1]$ with different items number truncated in Eq.(20). To keep the results' convergence, it can be seen that the truncated item number is 40, 70 and 100 at least respectively in Figs. 6, 7 and 8.

Table 1. The values of $a_i\ (i=5,6,\cdots,22)$ for FFW equation when $\alpha=0.4$

a_5	2	a_{11}	32	a_{17}	512
a_6	0	a_{12}	0	a_{18}	96
a_7	0	a_{13}	32	a_{19}	1024
a_8	8	a_{14}	128	a_{20}	2064
a_9	0	a_{15}	8	a_{21}	768
a_{10}	4	a_{16}	192	a_{22}	5120

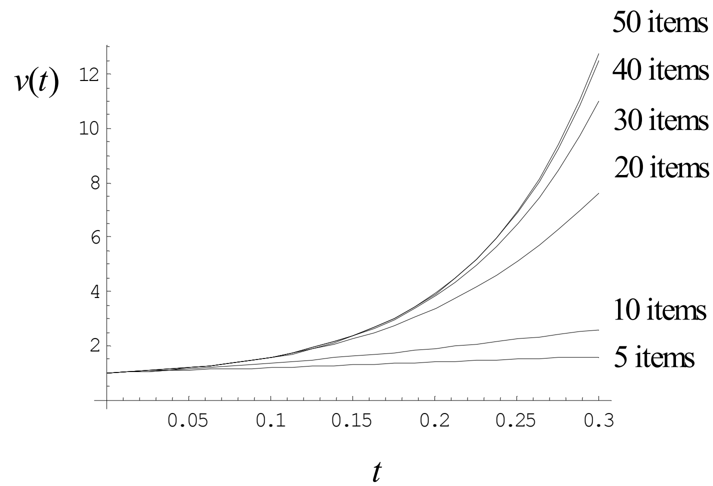

Fig. 6. v(t)'s curve by the undetermined coefficient method when $\alpha=0.4$, $t\in[0,0.3]$

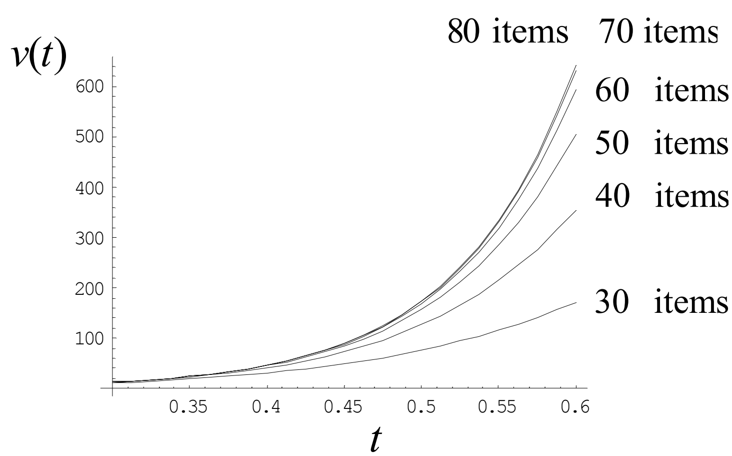

Fig. 7. v(t)'s curve by the undetermined coefficient method when $\alpha=0.4$, $t\in[0,0.6]$

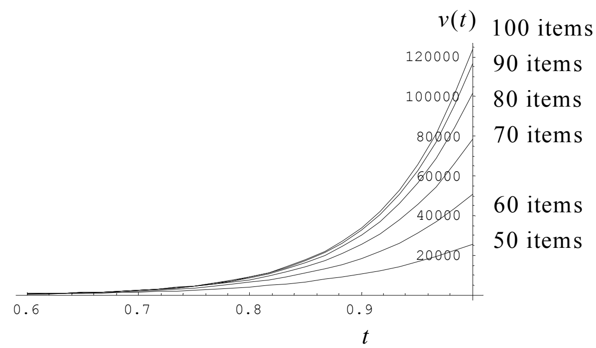

Fig. 8. v(t)'s curve by the undetermined coefficient method when $\alpha = 0.4$ **,** $t \in [0.6,1]$

4.3 Solution for $\alpha = 1.2$ by the undetermined coefficient method

When $\alpha = 1.2$, then m, n, θ in Eq.(20) are respectively $m = 6, n = 5$ and $\theta = 0.2$. As $m > n$, the initial values of a_i are obtained in section 4. as $a_0 = 1, a_i = 0, (i = 1, 2, \cdots, 5)$. From Eq.(31), the iteration relationship is established for the coefficients of a_i as

$$a_{6+i} = \frac{1}{4}\left(a_{5+i} - 2a_i\right) \quad (i = 0, 1, 2, \cdots) \tag{36}$$

The values of a_i $(i = 6, 7, \cdots, 18)$ are displayed in Tab. 2 after computation by an iteration program.

Table 2. The values of a_i $(i = 6, 7, \cdots, 18)$ **for FFW equation when** $\alpha = 1.2$

a_6	$-\frac{1}{2}$	a_{11}	$-\frac{1}{2048}$	a_{16}	$\frac{10239}{2097152}$
a_7	$-\frac{1}{8}$	a_{12}	$\frac{2047}{8192}$	a_{17}	$\frac{12287}{8388608}$
a_8	$-\frac{1}{32}$	a_{13}	$\frac{4095}{32768}$	a_{18}	$-\frac{4179969}{33554432}$
a_9	$-\frac{1}{128}$	a_{14}	$\frac{6143}{131072}$	a_{19}	$-\frac{12566529}{134217728}$
a_{10}	$-\frac{1}{512}$	a_{15}	$\frac{8191}{524288}$	a_{20}	$-\frac{25147393}{536870912}$

The expression of $v(t)$ in Eq. (20) is as follows

$$\begin{aligned} v(t) = 1 &- \frac{1}{2\Gamma(2.2)}t^{1.2} - \frac{1}{8\Gamma(2.4)}t^{1.4} - \frac{1}{32\Gamma(2.6)}t^{1.6} - \frac{1}{128\Gamma(2.8)}t^{1.8} \\ &- \frac{1}{512\Gamma(3)}t^{2} - \frac{1}{2048\Gamma(3.2)}t^{2.2} + \frac{2047}{8192\Gamma(3.4)}t^{2.4} + \cdots \end{aligned} \tag{37}$$

The approximate expression of Eq. (37) is

$$v(t) = 1 - 0.453802t^{1.2} - 0.10063t^{1.4} - 0.0218589t^{1.6} - 0.00466003t^{1.8} - 0.000976563t^{2} - 0.000201439t^{2.2} + 0.0838177t^{2.4} + \cdots \quad (38)$$

When $\alpha = 1.2$, Figs. 9-11 show v(t)'s curve by the undetermined coefficient method for three time intervals: $[0,2]$, $[4,6]$ and $[0,6]$. To keep the results' convergence, it is seen that the truncated item number is at least 20 and 70 respectively in Figs. 9 and 10.

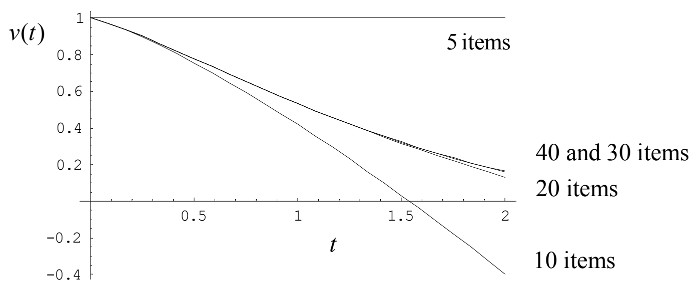

Fig. 9. v(t)'s curve by the undetermined coefficient method when $\alpha = 1.2$ **,** $t \in [0,2]$

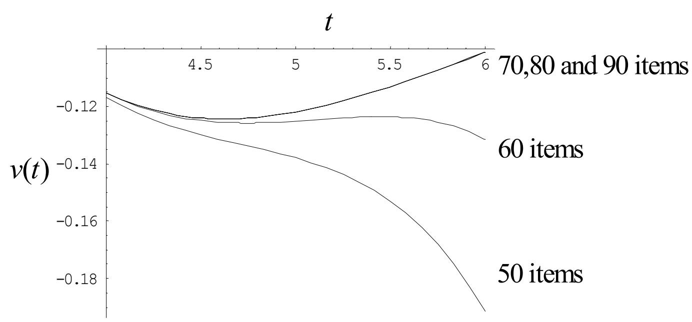

Fig. 10. v(t)'s curve by the undetermined coefficient method when $\alpha = 1.2$ **,** $t \in [4,6]$

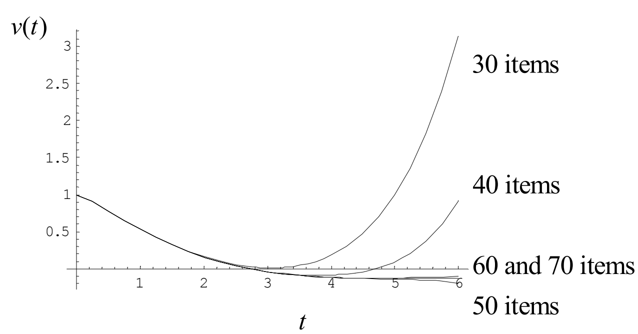

Fig. 11. v(t)'s curve by the undetermined coefficient method when $\alpha = 1.2$ **,** $t \in [0,6]$

4.4 Comparison of the solutions by the two methods

When $\alpha = 0.4$, comparing the expression of v_3 in Eq.(18) by FVIM with first several parts of Eq.(34) by using the undetermined coefficient method, the results are almost the same by the two methods. Figs. 2 and 8 are also almost the same.

For $\alpha = 1.2$, using the undetermined coefficient method, we have $\theta = 0.2$ and the result of $v(t)$ in Eq.(37) can be written as

$$\begin{aligned} v(t) = 1 + \frac{b_6}{\Gamma(1+6\theta)} t^{6\theta} + \frac{b_7}{\Gamma(1+7\theta)} t^{7\theta} + \frac{b_8}{\Gamma(1+8\theta)} t^{8\theta} \\ + \frac{b_{12}}{\Gamma(1+12\theta)} t^{12\theta} + \frac{b_{13}}{\Gamma(1+13\theta)} t^{13\theta} + \frac{b_{18}}{\Gamma(1+18\theta)} t^{18\theta} + \cdots \end{aligned} \tag{39}$$

where $b_6 = -1/2$, $b_7 = -1/8$, $b_8 = -1/32, b_{13} = 1/8, b_{18} = -1/8$. These coefficients are equal or near to the values of the coefficients $a_i, (i = 6,7,8,13,18)$ in Eq.（20） by FVIM from Table 2. According to Figs. 4 and 9, the curve of u(x,t)'s convergent numerical results are also almost the same . Thus the two proposed methods are both highly efficient for solving the FFW. The analytical solutions for the two cases are obtained respectively for the fractional order $0 < \alpha < 1$ and $1 \le \alpha < 2$.

5 Conclusion

In this study, the improved FVIM is presented where the Lagrange multiplier is determined by Wu. Comparing with the classical VIM, the modified version method is powerful to solve differential equations with more than two derivatives. The higher order approximate solutions of the FFW equation illustrate the method's efficiency and high accuracy compared with the solutions by the undetermined coefficient method.

The solutions of the FFW equation are totally different for the cases of the order of fractional derivative α between 1 and 2 and that between 0 and 1. When $1 \le \alpha < 2$, by using FVIM, the solution after several iterations is near to the accurate solutions, for example v_2,i.e. twice-iterated solution is converged for $\alpha = 1.2$ with time interval $[0, 2]$ in Fig. 4. While $0 < \alpha < 1$, the solution after at least 20 iterations is near to the accurate solutions when $\alpha = 0.4$ with time interval $[0, 0.6]$ in Fig. 1. The reason is that the value of $v(t)$ increases rapidly with time.

We point out an error in the existed literatures and we give a full conclusion with two cases of the fractional order's range. The existed results in some literatures in fact corresponds to the fractional order $1 \le \alpha < 2$, while the authors thought it as the result of the case of the fractional order less than 1.

With the exponential initial condition Eq. (13), no physical application gives such an exponential increase of the unknown with time in space at infinity as Figs. 2 or 8. So in practice, the parameter of the fractional order should be not less than 1.

Acknowledgements

The authors would like to express their deep gratitude to the referees for their valuable suggestions and comments. The work is financially supported by the NSFC (China, No11202146) and Qing Lan Project.

Competing Interests

Authors have declared that no competing interests exist.

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