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Filters and compactness on small categories and locales

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Abstract: In analogy with the classical theory of filters, for finitely complete or small categories, we provide the concepts of filter, $\mathfrak{G}$-neighborhood (short for "Grothendieck-neighborhood") and cover-neighborhood of points of such categories, to study convergence, cluster point, closure of sieves and compactness on objects of that kind of categories. Finally, we study all these concepts in the category **Loc** of locales.

Keywords: Filter; Bases of filters; Ultrafilter; S-filter; Cover-neighborhood; $\mathfrak{G}$-neighborhood; Grothendieck topology; Convergence; Compactness; Frames; Locales.

MSC: 18A35; 18F10; 18F70; 54A20; 54D30.

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1. Introduction

Convergence theory offers a versatile and effective framework for some areas of mathematics. Let us start by saying a few words about the history of this concept.

Convergence theory was probably defined firstly by Henri Cartan [1].

The notion of a limit along a filter was defined in his work in the maximum generality, as a filter on an arbitrary set and the limit defined for any map from this set to a topological space. However, the attention of mathematicians in the following years was mostly focused on two special cases.

- In general topology, the notion of the limit of a filter on a topological space X became one of the two primary tools used to describe the convergence together with the notion of a net. Some authors also studied the convergence of a sequence along with a filter.
- The definition of the limit along a filter can be reformulated using ideals- the dual notion of filter -. This type of limit of sequences was introduced independently by P. Kostyrko *et al.*, [2] and F. Nuray and W. H. Ruckle [3] and studied under the name " *I-convergence*". The motivation for this research direction was an effort to generalize some known results on statistical convergence.

In category theory, a sieve is choosing arrows with a common codomain. It is a categorical analog of a collection of open subsets of a fixed open set in topology. In a Grothendieck topology, certain sieves become categorical analogs of open covers in general topology.

In this paper, we use the concept of sieve to build filters in categories and locales; we explore the relationship between filters and Grothendieck topologies, defining the concept of $\mathfrak{G}$-convergence in order to carry out the study of compactness.

The paper is organized as follows: We describe, in §2, the notion of the sieve as in S. MacLane and I. Moerdijk [4]. In §3, we present the concepts of filters, filter base and we study the lattice structure of all filters on a category, and we present the concept of ultrafilter; after, in §4, we establish a connection between filters and Grothendieck topologies in the same category. In §5 we introduce the concepts of systems of the neighborhood, $\mathfrak{G}$-neighborhood of a point (recall that a point is an arrow with domain a terminal object), cover-neighborhood, convergence, cluster point, and closure of a sieve and some propositions about them. In §6 the notion of filter-preserving (or continuous) functor is presented; next, in §7 we use the convergence of ultrafilters in order to define compact objects in the categories in question. Finally, in §8, we study all the previous concepts in the category of locales.

2. Theoretical considerations

In the first part of this paper, we will work within an ambient category $\mathcal{C}$ which is finitely completed. Later we will do it in a small category.

From S. MacLane and I. Moerdijk [4], Chapter III, we have the following:

Let $\mathcal{C}$ be a category and let C be an object of $\mathcal{C}$. A sieve $\mathcal{S}$ on C is a family of morphisms in $\mathcal{C}$, all with codomain C, such that $f \in \mathcal{S} \Longrightarrow f \circ g \in \mathcal{S}$ Whenever this composition makes sense, in other words, $\mathcal{S}$ is the suitable ideal.

If $\mathcal{S}$ is a sieve on $\mathcal{C}$ and $h : D \to C$ is any arrow to C, then $h^*(\mathcal{S}) = \{g \mid cod(g) = D,\ h \circ g \in \mathcal{S}\}$ is a sieve on D.

The set $Sieve(C)$, of all sieves on C, is a partially ordered set under inclusion. It is easy to see that the union or intersection of any family of sieves on C is a sieve on C, so $Sieve(C)$ is a complete lattice.

3. Filters on a category

Definition 1. A filter on a category $\mathcal{C}$ is a function $\mathfrak{F}$ which assigns to each object C of $\mathcal{C}$ a collection $\mathfrak{F}(C)$ of sieves on C, in such a way that

(F_1) If $S \in \mathfrak{F}(C)$ and R is a sieve on C such that $S \subseteq R$, then $R \in \mathfrak{F}(C)$;
(F_2) every finite intersection of sieves of $\mathfrak{F}(C)$ belongs to $\mathfrak{F}(C)$;
(F_3) if $S \in \mathfrak{F}(C)$, then $h^*(S) \in \mathfrak{F}(D)$ for any arrow $h : D \to C$;
(F_4) the empty sieve is not in $\mathfrak{F}(C)$.

The pair $(C, \mathfrak{F}(C))$ will be called **a filtered object**.

Example 1. From the definition of a Grothendieck topology J on a category $\mathcal{C}$ it follows that for each object C of $\mathcal{C}$ and that

- for $S \in J(C)$ any larger sieve R on C is also a member of $J(C)$;
- it is also clear that if $R; S \in J(C)$ then $R \cap S \in J(C)$;
- consequently some Grothendieck topologies produce filters in the same category $\mathcal{C}$: they are exactly those for which $R \cap S \neq \emptyset$ for all pairs $R; S \in J(C)$ and such that the empty sieve is not in $J(C)$.
- Clearly, the **trivial topology** on $\mathcal{C}$ is a filter we shall call it **trivial filter**.
- It is also clear that the **atomic topology** on $\mathcal{C}$ (see [4]) is not a filter.

Remark 1. According to the previous example, for any site $(\mathcal{C}, J)$ there is a dense sub-site, given by the full subcategory of $\mathcal{C}$ on the objects which are not covered by the empty sieve with the induced topology, whose topos of sheaves is equivalent to Sh(C; J). This is an immediate application of the Comparison Lemma (see [5] Theorem 2.2.3).

Definition 2. A filter subbase on a category $\mathcal{C}$ is a function $\mathfrak{S}$ which assigns to each object C of $\mathcal{C}$ a collection $\mathfrak{S}(C)$ of sieves on $\mathcal{C}$, in such a way that no finite subcollection of $\mathfrak{S}(C)$ has an empty intersection.

An immediate consequence of this definition is

Proposition 1. *A sufficient condition that there should exist a filter $\mathfrak{S}'$ on a category $\mathcal{C}$ greater than or equal to a function $\mathfrak{S}$ (as above) is that $\mathfrak{S}$ should be a filter subbase on $\mathcal{C}$.*

Observe that $\mathfrak{S}'$ is the coarset filter greater than $\mathfrak{S}$.

Definition 3. A basis of a filter on a category $\mathcal{C}$ is a function $\mathfrak{B}$ which assigns to each object C of $\mathcal{C}$ and collection $\mathfrak{B}(C)$ of sieves on $\mathcal{C}$, in such a way that

(B_1) The intersection of two sieves of $\mathfrak{B}(C)$ contains a sieve of $\mathfrak{B}(C)$;
(B_2) if $\mathcal{S}$ is a sieve on $\mathfrak{B}$ and $h : D \to C$ is any arrow to C, then $h^*(\mathcal{S}) = \{g \mid cod(g) = D,\ h \circ g \in \mathcal{S}\}$ is a sieve on $\mathfrak{B}(D)$;
(B_3) $\mathfrak{B}(C)$ is not empty, and the empty sieve is not in $\mathfrak{B}(C)$.

Proposition 2. *If $\mathfrak{B}$ is a basis of filter on a category $\mathcal{C}$, then $\mathfrak{B}$ generates a filter $\mathfrak{F}$ by*

$$S \in \mathfrak{F}(C) \Leftrightarrow \exists R \in \mathfrak{B}(C) \text{ such that } R \subseteq S$$

for each object C of $\mathcal{C}$.

It is easy to check that this, indeed, defines a filter from a basis $\mathfrak{B}$.

3.1. The ordered set of all filters on a category

Definition 4. Given two filters $\mathfrak{F}_1$, $\mathfrak{F}_2$ on the same category $\mathcal{C}$, $\mathfrak{F}_2$ is said to be finer than $\mathfrak{F}_1$, or $\mathfrak{F}_1$ is coarser than $\mathfrak{F}_2$, if $\mathfrak{F}_1(C) \subseteq \mathfrak{F}_2(C)$ for all C object of $\mathcal{C}$.

In this way, the set of all filters on a category $\mathcal{C}$ is ordered by the relation **"$\mathfrak{F}_1$ is coarser than $\mathfrak{F}_2$"**.

Let $(\mathfrak{F}_i)_{i\in I}$ be a nonempty family of filters on a category $\mathcal{C}$; then the function $\mathfrak{F}$ which assigns to each object C the collection $\mathfrak{F}(C) = \bigcap_{i\in I} \mathfrak{F}_i(C)$ is manifestly a filter on $\mathcal{C}$ and is obviously the greatest lower bound of the family $(\mathfrak{F}_i)_{i\in I}$ on the ordered set of all filters on $\mathcal{C}$.

Definition 5. An ultrafilter on a category $\mathcal{C}$ is a filter such that there is no filter on $\mathcal{C}$ which is strictly finer than $\mathfrak{U}$.

Using the Zorn lemma, we deduce that

Proposition 3. *If $\mathfrak{F}$ is any filter on a category $\mathcal{C}$, there is an ultrafilter finer than $\mathfrak{F}$ on $\mathcal{C}$.*

Proposition 4. *Let $\mathfrak{U}$ be an ultrafilter on a category $\mathcal{C}$, and let C be an object of $\mathcal{C}$. Let S, T be sieves on C such that $S \cup T \in \mathfrak{U}(C)$ then either $S \in \mathfrak{U}(C)$ or $T \in \mathfrak{U}(C)$.*

Proof. If the affirmation is false, there exist sieves S, T on C that do not belong to $\mathfrak{U}(C)$, but $S \cup T \in \mathfrak{U}(C)$. Consider a function $\mathfrak{T} : \mathcal{C} \to Sets$ defined by $\mathfrak{T}(C) = \{R \in \ Sieve(C) \mid R \cup S \in \mathfrak{U}(C)\}$. Let us verify that $\mathfrak{T}$ is a filter on a $\mathcal{C}$: in fact, for any object C of the category $\mathcal{C}$, we have

- (F_1) if $R^{'} \in \mathfrak{T}(C)$ then $R^{'} \cup S \in \mathfrak{U}(C)$; and if $R^{''}$ is a sieve on C such that $R' \subseteq R''$, then $R^{''} \cup S \in \mathfrak{U}(C)$. Consequently $R^{''} \in \mathfrak{T}(C)$.
- (F_2) We must show that every finite intersection of sieves of $\mathfrak{T}(C)$ belongs to $\mathfrak{T}(C)$; indeed, let $(R_i)_{i=1,\cdots,n}$ be a finite collection of sieves on C such that $R_i \cup S \in \mathfrak{U}(C)$ for all $i = 1...n$, then $(R_1 \cup S) \cap (R_2 \cup S) \cap \cdots \cap (R_n \cup S) = (\bigcap_{i=1}^n R_i) \cup S \in \mathfrak{U}(C)$. which is equivalent to saying that $(\bigcap_{i=1}^n R_i) \in \mathfrak{T}(C)$.
- (F_3) If $R^{'} \in \mathfrak{T}(C)$ then $R^{'} \cup S \in \mathfrak{U}(C)$; and $h^*(R^{'} \cup S) \in \mathfrak{T}(D)$ for any arrow $h : D \to C$; in other words, $h^*(R^{'}) \cup h^*(S) \in \mathfrak{T}(D)$, therefore $h^*(R^{'}) \in \mathfrak{T}(D)$.
- (F_4) Evidently, the empty sieve is not in $\mathfrak{T}(C)$.

Therefore $\mathfrak{T}$ is a filter finer than $\mathfrak{U}$, since $T \in \mathfrak{T}(C)$; but this contradicts the hypothesis than $\mathfrak{U}$ is an ultrafilter. □

Corollary 1. *If the union of a finite sequence $(S_i)_{i=1,\cdots,n}$ of sieves on C belongs to the image $\mathfrak{U}(C)$ under an ultrafilter $\mathfrak{U}$, then at least one of the S_i belongs to $\mathfrak{U}(C)$.*

Proof. The proof is a simple use of induction on n. □

4. Filters and Grothendieck topologies

Our main aim in this section is to establish some connections between filters and Grothendieck topologies in the same category.

First we need some observations.

Definition 6.

1. If J_1 and J_2 are Grothendieck topologies on a category $\mathcal{C}$, we say that $J_1 \preceq J_2$ if and only if $J_1(C) \subseteq J_2(C)$ for all objects C of $\mathcal{C}$.
2. In the same way, if $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are filters on a category $\mathcal{C}$, we say that $\mathfrak{F}_1 \preceq \mathfrak{F}_2$ if and only if $\mathfrak{F}_1(C) \subseteq \mathfrak{F}_2(C)$ for all objects C of $\mathcal{C}$.

It is easy to verify that this definition produces two order relations on Grothendieck topologies and filters respectively.

In this way, we have the following facts:

Lemma 1. *Every filter on a category is a Grothendieck topology on the same category.*

Proof.

- Given a filter $\mathfrak{F}$ and an object C of $\mathcal{C}$, suposse that S is a sieve on C. Since $S \subseteq t_C$, we have that $t_C \in \mathfrak{F}(C)$.
- If $S \in \mathfrak{F}(C)$, then certainly $h^*(S) \in \mathfrak{F}(D)$ for any arrow $h : D \to C$.
- If $S \in \mathfrak{F}(C)$ and R is any sieve on C such that $h^*(S) \in \mathfrak{F}(D)$ for any arrow $h : D \to C$ in S, then $h^*(S \cap R) = h^*(S) \cap h^*(R) \in \mathfrak{F}(D)$, consequently $S \cap R \in \mathfrak{F}(C)$, and since $S \cap R \subseteq R$ we have $R \in \mathfrak{F}(C)$.

□

Lemma 2. *Let $\mathfrak{F}$ be a filter on a category $\mathcal{C}$ and let J be a Grothendieck topology on the same category. If $J \preceq \mathfrak{F}$ then J is a filter.*

Proof.

(F_1) Given any object C of $\mathcal{C}$, it is clear that if S is a sieve in $J(C)$ and R is a sieve on C such that $S \subseteq R$, then $R \in J(C)$;
(F_2) let $(R_i)_{i=1,\cdots,n}$ be a finite collection of sieves on $J(C)$, then $\bigcap_{i=1}^n R_i \in J(C)$, and therefore $\bigcap_{i=1}^n R_i \in \mathfrak{F}(C)$ (and consequently is not empty);
(F_3) If $S \in J(C)$, then certainly $h^*(S) \in J(D)$ for any arrow $h : D \to C$.
(F_4) the empty sieve is neither in $\mathfrak{F}(C)$ nor in $J(C)$.

□

4.1. Product of filters

Finally in this section we shall be interested in studying a category $\mathcal{C}$ equipped with a family $(\mathfrak{F}_i)_{i\in I}$ of filters.

Proposition 5. *Let $\mathcal{C}$ be a category equipped with a family $(\mathfrak{F}_i)_{i\in I}$ of filters, and let $(C_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. Then the function $\mathfrak{B}$ which assigns to each object $C = \prod_{i\in I} C_i$, the collection of sieves $\mathfrak{B}(C) = \left\{\prod_{i\in I} S_i \mid S_i \in \mathfrak{F}_i(C)\right\}$, where $S_i = t_{C_i}$ is the maximal sieve on C_i except for a finite number of indices, is basis of a filter on $\mathcal{C}$.*

Proof.

(B_1) The formula $\left(\prod_{i\in I} S_i\right) \cap \left(\prod_{i\in I} T_i\right) = \prod_{i\in I}(S_i \cap T_i)$ ensure that the intersection of two sieves on $\mathfrak{B}(C)$ contains a sieve of $\mathfrak{B}(C)$;
(B_2) if $C = \prod_{i\in I} C_i$ and $D = \prod_{i\in I} D_i$ are objects of $\mathcal{C}$ and $h : D \to C$ is any arrow to C, since limits commute with limits, we have $h^*\left(\prod_{i\in I} S_i\right) = \prod_{i\in I} S_i(h^*(S_i))$. Therefore $h^*\left(\prod_{i\in I} S_i\right)$ is a sieve on $\mathfrak{B}(D)$.
(B_3) This last assertion is immediate from the first.

□

Corollary 2. *The filter of base $\mathfrak{B}$ which assigns to each object $C = \prod_{i\in I} C_i$, the collection of sieves $\mathfrak{B}(C) = \left\{\prod_{i\in I} S_i \mid S_i \in \mathfrak{F}_i(C)\right\}$, where $S_i = t_{C_i}$ is the maximal sieve on C_i except for a finite number of indices, is basis of a filter on $\mathcal{C}$, is also generated by the sets ${pr_i}^{-1}(S_i)$, where S_i is a sieve on C_i and i runs through I.*

Proof. It is a consequence of the fact that ${pr_i}^{-1}(S_i) = S_i \times \prod_{j\neq i} t_{C_j}$. □

5. Systems of Neighborhoods

Recall that a **point** of an object C of a category $\mathcal{C}$ is a morphism $p : 1 \to C$, where 1 is a terminal object of $\mathcal{C}$.

Definition 7. Let $(\mathcal{C}, J)$ be a category equipped with a Grothendieck topology, and let C be an object of $\mathcal{C}$. A sieve V in $J(C)$, is said to be a $\mathfrak{G}$-neighborhood of a point $p : 1 \to C$ if there exist a morphism $\phi : D \to C$ in V and a point $q : 1 \to D$ such that $\phi \circ q = p$.

Definition 8. Let $(\mathcal{C}, J)$ be a category equipped with a Grothendieck topology. A cover-neighborhood of $(\mathcal{C}, J)$ is a function $\mathcal{N}$ which assigns to each object $(C, J(C))$ of $(\mathcal{C}, J)$ and to each point $p_C : 1 \to C$, a collection $\mathcal{N}_{p_C}(C)$ of sieves of $\mathcal{C}$ such that each sieve in $\mathcal{N}_{p_C}(C)$ contains a $\mathfrak{G}$-neighborhood of p_C.

Proposition 6. *Let $\mathcal{C}$ be a category, and let C be an object of $\mathcal{C}$. The pair $(C, \mathcal{N}_p(C))$, where $\mathcal{N}_p(C)$ is the collection of all cover-neighborhoods of a point $p : 1 \to C$, is **a filtered object***

Proof.

(i) If $S \in \mathcal{N}_p(C)$ and R is a sieve on C such that $S \subseteq R$, then $R \in \mathcal{N}_p(C)$, because there is a $\mathfrak{G}$-neighborhood V of p_C such that $V \subseteq S \subseteq R$;

(ii) let $\{S_1, S_2, \cdots, S_n\}$ be a finite collection of sieves of $\mathcal{N}_p(C)$, then there exists a collection $\{V_1, V_2, \cdots, V_n\}$ of $\mathfrak{G}$-neighborhood of p_C such that $V_i \subseteq S_i$ for $I = 1, 2, \cdots n$, therefore $\bigcap_{i=1}^{n} V_n \subseteq \bigcap_{i=1}^{n} S_n$ and $\bigcap_{i=1}^{n} S_n \in \mathcal{N}_p(C)$;

(iii) the empty sieve is not in $\mathcal{N}_p(C)$ (each sieve contains a point $p : 1 \to C$).

□

In this case, we say that the point $p : 1 \to C$ is a **limit point** of $\mathcal{N}_p(C)$.

Definition 9. Let $(\mathcal{C}, J)$ be a category equipped with a Grothendieck topology; let $\mathfrak{F}$ be a filter on $\mathcal{C}$ and let C be an object of $\mathcal{C}$.

1. We shall say that $\mathfrak{F}(C)$ **converges** to a point $p : 1 \to C$ if $\mathcal{N}_p(C) \subseteq \mathfrak{F}(C)$.
2. The closure of a sieve A on C is the collection of all points $p : 1 \to C$ such that every cover-neighborhood of p meets A.
3. A point $p : 1 \to C$ is a cluster point of $\mathfrak{B}(C)$ -the image under the filter base $\mathfrak{B}$ of C- if it lies in the closure of all the sieves on $\mathfrak{B}(C)$.
4. A point $p : 1 \to C$ is a cluster point of $\mathfrak{F}(C)$ -the image under the filter $\mathfrak{F}$ of C- if it lies in the closure of all the sieves on $\mathfrak{F}(C)$.
5. When $\mathfrak{F}(C)$ converges to a point $p : 1 \to C$, we shall say that $p : 1 \to C$ is a **limit** point of $\mathfrak{F}(C)$.

Definition 10. Let $(\mathcal{C}, J)$ be a category equipped with a Grothendieck topology; let $\mathfrak{B}$ a basis of a filter on $\mathcal{C}$ and let C be an object of $\mathcal{C}$. The point $p : 1 \to C$ is said to be a limit of $\mathfrak{B}(C)$ if the image of C, by the filter whose base is $\mathfrak{B}$, converges to $p : 1 \to C$.

Proposition 7. *Let $(\mathcal{C}, J)$ be a category equipped with a Grothendieck topology; let $\mathfrak{F}$ be a filter on $\mathcal{C}$ and let C be an object of $\mathcal{C}$. The point $p : 1 \to C$ is a cluster point of $\mathfrak{F}(C)$ if and only if there exists a filter $\mathcal{G}$ finer than $\mathfrak{F}$ such that $\mathcal{G}(C)$ **converges** to $p : 1 \to C$.*

Proof. Let us begin by assuming that the point $p : 1 \to C$ is a cluster point of $\mathfrak{F}(C)$; from Definition 9, it follows that for each sieve A in $\mathfrak{F}(C)$, every $\mathfrak{G}$-neighborhood V of p meets A. We need to show that the collection $\mathcal{B}(C) = \{A \cap V \mid V \text{ is a } \mathfrak{G}\text{-neighborhood of } p\}$ define a base for a filter $\mathcal{G}$ finer than $\mathfrak{F}$.in such a way that $\mathcal{G}(C)$ **converges** to $p : 1 \to C$.

Indeed,

(B_1) Let $A \cap V$, $A \cup W$ two elements of the collection $\mathcal{B}(C)$, since $(A \cap V) \cap (A \cup W) = A \cup (V \cap W)$ and $V \cap W$ is a $\mathfrak{G}$-neighborhood of p, there exists U, a $\mathfrak{G}$-neighborhood of p such that $U \subseteq V \cap W$, and clearly $A \cap U \in \mathcal{B}(C)$;

(B_2) Obviously $\mathcal{B}(C)$ is not empty, and the empty sieve is not in $\mathcal{B}(C)$.

Now, if $\mathcal{G}$ is the filter generated by $\mathcal{B}$ then $\mathcal{G}$ is finer than $\mathfrak{F}$, and $\mathcal{G}(C)$ naturally converges to $p : 1 \to C$.

Conversely, if there is a filter $\mathcal{G}$ finer than $\mathfrak{F}$ such that $\mathcal{G}(C)$ **converges** to $p : 1 \to C$ then each sieve R in $\mathfrak{F}(C)$ and each $\mathfrak{G}$-neighborhood U of $p : 1 \to C$ belongs to $\mathcal{G}$ and hence meet, so the point $p : 1 \to C$ is a cluster point of $\mathfrak{F}(C)$. $\square$

Proposition 8. *Let $(\mathcal{C}, J)$ be a category equipped with a Grothendieck topology; let C be an object of $\mathcal{C}$ and let A be a sieve on C. The point $p : 1 \to C$ lies in the closure of A if and only if there is a filter $\mathcal{G}$ such that $A \in \mathcal{G}(C)$ and $\mathcal{G}(C)$* ***converges*** *to $p : 1 \to C$.*

Proof. Let us begin by assuming that the point $p : 1 \to C$ lies in the closure of A; from Definition 9, it follows that every $\mathfrak{G}$-neighborhood V of p meets A. Then $\mathcal{B}(C) = \{A \cap V \mid V \text{ is a } \mathfrak{G}\text{-neighborhood of } p\}$ is a base for a filter $\mathcal{G}$, in such a way that $\mathcal{G}(C)$ **converges** to $p : 1 \to C$.

Conversely, if $A \in \mathcal{G}(C)$ and $\mathcal{G}(C)$ **converges** to $p : 1 \to C$ then $p : 1 \to C$ is a cluster point of $\mathcal{G}(C)$ and hence $p : 1 \to C$ lies in the closure of A. $\square$

Corollary 3. *Let $\mathfrak{U}$ be an ultrafilter on a category $\mathcal{C}$, and let C be an object of $\mathcal{C}$. $\mathfrak{U}(C)$ converges to a point $p : 1 \to C$ if and only if $p : 1 \to C$ is a cluster point of $\mathfrak{U}(C)$.*

Example 2. Let $\mathcal{A}$ be a complete Heyting algebra and regard $\mathcal{A}$ as a category in the usual way.

- Then (see [4]) $\mathcal{A}$ can be equipped with a base for a Grothendieck topology K, given by $\{a_i \mid i \in I\} \in K(c)$ if and only if $\bigvee_{i \in I} = c$, where $\{a_i \mid i \in I\} \subseteq \mathcal{A}$ and $c \in \mathcal{A}$.
- A sieve S on an element c of $\mathcal{A}$ is just a subset of elements $b \leqslant c$ such that $a \leqslant b \in S$ implies $a \in S$.
- In the Grothendieck topology J with basis K, a sieve S on c covers c iff $\bigvee S = c$.
- A filter on $\mathcal{A}$ is a function $\mathfrak{F}$ which assigns to each element c of $\mathcal{A}$ a collection $\mathfrak{F}(c)$ of sieves, such that

  (F_1) If $S \in \mathfrak{F}(c)$ and R is a sieve on c such that $S \subseteq R$, then $R \in \mathfrak{F}(c)$;
  (F_2) every finite intersection of sieves of $\mathfrak{F}(c)$ belongs to $\mathfrak{F}(c)$;
  (F_3) the empty sieve is not in $\mathfrak{F}(c)$.

- An immediate consequence of the previous construction of a Grothendieck topology and a filter on $\mathcal{A}$ is that $\mathfrak{F}(c)$ converges to c iff $\bigvee S = c$, for each $S \in \mathfrak{F}(c)$.

6. Filter-preserving functors

Definition 11. Let $(\mathcal{C}, \mathfrak{F})$ and $(\mathcal{D}, \mathfrak{G})$ be small categories equipped with filters and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. We say F is filter-preserving (or continuous) if, for any $c \in ob(\mathcal{C})$ and any covering sieve $R \in \mathfrak{F}(c)$, the family $\{F(f) \mid f \in R\}$ generates a covering sieve $S \in \mathfrak{G}(F(c))$, consisting of all the morphisms with codomain $F(c)$ which factor through at least one $F(f)$.

We shall use the notation $\langle F(R) \rangle$ to denote the covering sieve generated by the family $\{F(f) \mid f \in R\}$ in $\mathfrak{G}(F(c))$.

Proposition 9. *Let $F : (\mathcal{C}, J) \longrightarrow (\mathcal{D}, K)$ be a morphism of sites. If, for every object C of $\mathcal{C}$, V is a $\mathfrak{G}$-neighborhood of a point $p : 1 \to C$, then the family $F(V) = \{F(\alpha) \mid \alpha \in V\}$ generates a $\mathfrak{G}$-neighborhood $\langle F(V) \rangle$ of the point $F(p)$ of $F(C)$ in $\mathcal{D}$.*

Proof. The hypothesis that V is a $\mathfrak{G}$-neighborhood of a point $p : 1 \to C$ tell us that there exists a morphism $\phi : D \to C$ in V and a point $q : 1 \to D$ such that $\phi \circ q = p$.

Next, we apply functor F to obtain $F(\phi) \circ F(q) = F(p)$, where , of course, $F(\phi)$ is in the $\mathfrak{G}$-neighborhood $\langle F(V) \rangle$ of the point $F(p)$. $\square$

Proposition 10. *Let $F : (\mathcal{C}, J) \longrightarrow (\mathcal{D}, K)$ be a morphism of sites and let $\mathcal{N}$ be a cover-neighborhood of $(\mathcal{C}, J)$ then $\langle F(\mathcal{N}) \rangle$ is a a cover-neighborhood of $(\mathcal{D}, K)$.*

Proof. Let V in $J(C)$, be a $\mathfrak{G}$-neighborhood of a point $p_C :\to C$, and let W a sieve in $\mathcal{N}_{p_C}(C)$, where $\mathcal{N}$ is a cover-neighborhood of $(\mathcal{C}, J)$, such that $V \hookrightarrow W$ is an inclusion. Since F is a morphisms of sites, $F(V) \hookrightarrow F(W)$ is also an inclusion which belongs to $\langle \mathcal{N}_{p_C}(C) \rangle$, and clearly $\langle F(\mathcal{N}) \rangle$ is a a cover-neighborhood of $(\mathcal{D}, K)$.
$\square$

Proposition 11. *Let $F : (\mathcal{C}, J) \longrightarrow (\mathcal{D}, K)$ be a morphism of sites and let $\mathfrak{B}$ be a basis of a filter on the category $\mathcal{C}$ then $F(\mathfrak{B}(C))$ is a basis of filter on the category $\mathcal{D}$.*

Proof. let C be an object of $\mathcal{C}$ We shall show that

- (B_1) The intersection of two sieves on $F(\mathfrak{B}(C))$ contains a sieve on $F(\mathfrak{B}(C))$;
- (B_2) If S is a sieve on $F(\mathfrak{B}(C))$ and $h : F(D) \to F(C)$ is any arrow to $F(C)$, then $\big(F(h)\big)^*\big(F(S)\big)$ is a sieve on $F\big(\mathfrak{B}(D)\big)$.
- (B_3) $F(\mathfrak{B}(C))$ is not empty, and the empty sieve is not on $F(\mathfrak{B}(C))$.

First, since a morphisms of sites preserves inclusion, for sieves S, T on $\mathfrak{B}(C)$, we have $F(S \cap T) \subseteq F(S) \cap F(T)$.

Next, let S be a sieve on $\mathfrak{B}(C)$ and let $h : D \to C$ be any arrow to C such that $h^*(S)$ is a sieve on $\mathfrak{B}(D))$, then $F(S)$ is a sieve on $F\big(\mathfrak{B}(C)\big)$ and for $F(h) : F(D) \to F(C)$, we have that $F(h)^*\big(F(S)\big)$ is a sieve on $F\big(\mathfrak{B}(D)\big)$.

Finally, for every sieve S on $\mathfrak{B}(C)$, the fact that $S \neq \emptyset$ implies $F(S) \neq \emptyset$. □

7. Compactness on small categories

Definition 12. Let $(\mathcal{C}, J)$ be a site and let C be an object of $\mathcal{C}$. We say that an object C of $\mathcal{C}$

- Is **quasi-compact** if, for every filter $\mathfrak{F}$ on $\mathcal{C}$, $\mathfrak{F}(C)$ has at least one cluster point.
- Is **Hausdorff** if, for every filter $\mathfrak{F}$ on $\mathcal{C}$, $\mathfrak{F}(C)$ has no more that one limit point.
- Is **compact** if, for every filter $\mathfrak{F}$ on $\mathcal{C}$, $\mathfrak{F}(C)$ is quasi-compact and Hausdorff.

Lemma 3. *Let $(\mathcal{C}, J)$ be a site and let C be an object of $\mathcal{C}$. An object C of $\mathcal{C}$ is compact if and only if, for every ultrafilter $\mathfrak{U}$ on $\mathcal{C}$, $\mathfrak{U}(C)$ is convergent.*

Proof. First suppose that $\mathfrak{F}$ is a filter on $\mathcal{C}$. Proposition 3 ensures that, for every filter, there exists an ultrafilter $\mathfrak{U}$ finer than $\mathfrak{F}$, such that $\mathfrak{U}(C)$ converges to a point p on C, therefore p is a cluster point of $\mathfrak{F}(C)$.

Conversely, if, for an ultrafilter $\mathfrak{U}$, $\mathfrak{U}(C)$ has a cluster point then it converges to this point. □

Proposition 12. *Let $(\mathcal{C}, \mathfrak{F})$ and $(\mathcal{D}, \mathfrak{G})$ be small categories equipped with filters and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a filter-preserving (continuous) functor. If C is a compact object of $\mathcal{C}$, then $\mathfrak{F}(C)$ is a compact object of $\mathcal{D}$.*

Proof. This is a consequence of Propositions 9, 10, 11 and Lemma 11. □

Theorem 4. *(Tychonoff) Let $(\mathcal{C}, J)$ be a site. Every product of compact objects in the category $\mathcal{C}$ is compact.*

Proof. Suppose we have chosen a collection $(C_i)_{i\in I}$ of compact objects in $\mathcal{C}$; equivalently, for every family $(\mathfrak{U}_i)_{i\in I}$ of ultrafilters on $\mathcal{C}$, $\mathfrak{U}_i(C_i)$ is convergent. Then the function $\mathfrak{B}$ which assigns to each object $C = \prod\limits_{i\in I} C_i$, the collection of sieves $\mathfrak{B}(C) = \Big\{\prod\limits_{i\in I} S_i \mid S_i \in \mathfrak{F}_i(C)\Big\}$, where $S_i = t_{C_i}$ is the maximal sieve on C_i except for a finite number of indices, is basis of an ultrafilter on $\mathcal{C}$. □

8. Filters and compactness on locales

There is a significant 'generalization' of the notion of topological space, namely the notion of locale. The object of this section is to present an alternative approach to the notion of compactness on locales, different from (but not entirely independent of) the one which we have followed previously.

From P.J. Johnstone ([5]) and A. J. Lindenhovius ([6]), we take the following ideas

Definition 13.

1. The category $\mathbf{F}rm$ of frames is the category whose objects are complete lattices satisfying the infinite distributive law, and whose morphisms are functions preserving finite meets and arbitrary joins.
2. The category $\mathbf{L}oc$ of locales is the opposite of the category $\mathbf{F}rm$. We refer to morphisms in $\mathbf{L}oc$ as continuous maps, and write Ω for the functor $\mathbf{S}p \to \mathbf{L}oc$ which sends a topological space to its lattice of open sets, and a continuous map $f : X \to Y$ to the function $f^{-1} : \Omega(Y) \to \Omega(X)$.

Definition 14. Let $(P, \leqslant)$ be a lattice and $M \subseteq P$. We say that

- $a \in P$ is a join (or least upper bound) for M, and write $a = \bigvee M$ if

    1. a is an upper bound for M, i.e. $m \leqslant a$ for all $m \in M$, and
    2. if b satisfies for all $m \in M$ $(m \leqslant b)$ then $a \leqslant b$.

- If M is a two-element set $\{m, k\}$, we write $m \vee k$ for $\bigvee\{m, k\}$
- if M is the empty set $\emptyset$, we write 0 for $\bigvee \emptyset$, clearly 0 is just the least element of M.
- Dually, in any lattice we can consider the notion of meet (greatest lower bound), defined by reversing all the inequalities in the definition of join. We write $\bigwedge M$, $a \wedge b$ and 1 for the analogues of $\bigvee M$, $a \vee b$ and 0.
- M is an up-set if for each $x \in M$ and $y \in P$ we have $x \leqslant y$ implies $y \in M$.
- Similarly, M is called a down-set if for each $x \in M$ and $y \in P$ we have $y \leqslant x$ implies $y \in M$.
- Given an element $x \in P$, we define the up-set and down-set generated by x by $\uparrow x = \{y \in P : x \leqslant y\}$ and $\downarrow x = \{y \in P : y \leqslant x\}$, respectively.
- We can also define the up-set generated by a subset M of P by $\uparrow M = \{x \in P : m \leqslant x \text{ for some } m \in M\} = \bigcup_{m \in M} \uparrow m$, and similarly, we define the down-set generated by M by $\downarrow M = \bigcup_{m \in M} \downarrow m$.

Notation 1. We denote the collection of all up-sets of a partially ordered set P by $\mathcal{U}(P)$ and the set of all down-sets by $\mathcal{D}(P)$.

Example 3. For $Z_D^+ := (\mathbb{Z}^+, \leqslant_|)$, where $\leqslant_|$ is the multiplicative (or divisibility) partial order on the set $\mathbb{Z}^+$ of positive integers, we can observe that

- The down-set generated by an element n of $\mathbb{Z}^+$ is the collection $\downarrow n = \{k \in \mathbb{Z}^+ : k \leqslant_| n\} = \mathfrak{D}_n$, the set of all divisors of n.
- If $M \subseteq \mathbb{Z}^+$ then the down-set generated by M is $\downarrow M = \bigcup_{m \in M} \mathfrak{D}_m$.
- The up-set generated by $n \in \mathbb{Z}^+$ is $\uparrow n = \{k \in \mathbb{Z}^+ : n \leqslant_| k\}$, i.e. the set $\mathfrak{M}_n$ of all multiples of n.
- If $P \subseteq \mathbb{Z}^+$ then the up-set generated by P is $\uparrow M = \bigcup_{p \in P} \mathfrak{M}_p$.
- An interesting fact about $\mathfrak{D}_n$, the set of all divisors of n, is that it is a **locale**.
- Using the Fundamental Theorem of Arithmetic it is easy to show that if $n \in \mathbb{Z}^+$ is the product of a finite number of distinct primes then $\mathfrak{D}_n$ is a Boolean Algebra.

8.1. Grothendieck topologies on locales

In order to construct Grothendieck topologies, we first define sieves.

Definition 15. Given an element k in a locale L, a subset S of L is called a sieve on k if $S \in \mathcal{D}(\downarrow k)$.

Definition 16. A Grothendieck topology on a locale L is a function J which assigns to each object k of L a collection $J(k)$ of sieves on k, in such a way that

(i) the maximal sieve $\downarrow k$ is in $J(k)$;
(ii) (stability axiom) if $S \in J(k)$ and $m \leqslant k$ then $S \cap (\downarrow m)$ is in $J(m)$;
(iii) (transitivity axiom) if $S \in J(k)$ and R is any sieve on k such that $R \cap (\downarrow m)$ is in $J(m)$ for each $m \in S$, then $R \in J(k)$.

Example 4.

- The trivial Grothendieck topology on L is given by $J_{tri}(n) = \downarrow n$.
- The discrete Grothendieck topology on the lattice L is given by $J_{dis}(n) = \mathcal{D}(\downarrow n)$.
- The atomic Grothendieck topology on L can only be defined if L is downwards directed, and is given by $J_{atom}(n) = \mathcal{D}(\downarrow n) - \{\emptyset\}$.
- A subset $D \subseteq (\downarrow n)$ is said to be dense below n if for any $m \leqslant n$ there exists $k \leqslant m$ with $k \in D$.
- The dense topology on L is given by

$$J(n) = \{D \mid k \leqslant n \text{ for all } k \in D, \text{ and } D \text{ is a sieve dense below } n\}.$$

8.2. S-Filters on locales

Recall that a basic notion in complete lattices is that of **filter**: a filter F of L is a non-empty subset of L such that

1. F is a sublattice of L , and
2. for any $a \in F$ and $b \in L$, $a \vee b \in F$.

In a different direction, we have the notion of **filter of sieves (ideals)** on a locale L, which we shall call S-**filter**.

Definition 17. An S-filter on a locale L is a function $\mathfrak{F}$ which assigns to each object k of L a collection $\mathfrak{F}(k)$ of sieves on k, in such a way that

(F_1) if $S \in \mathfrak{F}(k)$ and R is a sieve on k such that $S \subseteq R$, then $R \in \mathfrak{F}(k)$;
(F_2) every finite intersection of sieves of $\mathfrak{F}(k)$ belongs to $\mathfrak{F}(k)$;
(F_3) if $S \in J(k)$ and $m \leqslant k$ then $S \cap (\downarrow m)$ is in $\mathfrak{F}(m)$;
(F_4) the empty sieve is not in $\mathfrak{F}(k)$.

Example 5. From the definition of a Grothendieck topology J on a locale L it follows that for each object k of L and that

- for $S \in J(k)$ any larger sieve R on C is also a member of $J(k)$.
- It is also clear that if $R; S \in J(k)$ then $R \cap S \in J(k)$,
- consequently some Grothendieck topologies produce S-filters in the same locale L: they are exactly those for which
$$R \cap S \neq \emptyset \quad \text{for all pairs} \quad R; S \in J(k) \tag{1}$$
and such that the empty sieve is not in $J(k)$.
- Clearly, the **trivial topology** on k is a S-filter we shall call it **trivial** S-**filter**.
- It is also clear that the **atomic topology** on L (see [4]) is not a S-filter.

Definition 18. A basis of an S-filter on a locale L is a function $\mathfrak{B}$ which assigns to each $k \in L$ a collection $\mathfrak{B}(k)$ of sieves on k, in such a way that

(B_1) The intersection of two sieves of $\mathfrak{B}(k)$ contains a sieve of $\mathfrak{B}(k)$;
(B_2) if S is a sieve on $\mathfrak{B}(k)$ and $m \leqslant k$ then $S \cap (\downarrow m)$ is in $\mathfrak{B}(m)$;
(B_3) $\mathfrak{B}(k)$ is not empty and the empty sieve is not in $\mathfrak{B}(k)$.

Proposition 13. *If $\mathfrak{B}$ is a basis of S-filter on a locale L, then $\mathfrak{B}$ generates an S-filter $\mathfrak{F}$ by $S \in \mathfrak{F}(k) \Leftrightarrow \exists R \in \mathfrak{B}(k)$ such that $R \subseteq S$ for each object $k \in L$.*

It is easy to check that this, indeed, defines a S-filter from a basis $\mathfrak{B}$.

8.3. The ordered set of all S-filters on a locale

Definition 19. Given two S-filters $\mathfrak{F}_1$, $\mathfrak{F}_2$ on the same locale L, $\mathfrak{F}_2$ is said to be finer than $\mathfrak{F}_1$, or $\mathfrak{F}_1$ is coarser than $\mathfrak{F}_2$, if $\mathfrak{F}_1(k) \subseteq \mathfrak{F}_2(k)$ for all $k \in L$.

In this way, the set of all S-filters on a locale L is ordered by the relation **"$\mathfrak{F}_1$ is coarser than $\mathfrak{F}_2$"**.

Let $(\mathfrak{F}_i)_{i \in I}$ be a nonempty family of S-filters on a L; then the function $\mathfrak{F}$ which assigns to each object $k \in L$ the collection $\mathfrak{F}(k) = \bigcap_{i \in I} \mathfrak{F}_i(k)$ is manifestly a S-filter on L and is obviously the greatest lower bound of the family $(\mathfrak{F}_i)_{i \in I}$ on the ordered set of all S-filters on L.

Definition 20. An S-ultrafilter on a locale L is a S-filter $\mathfrak{U}$ such that there is no S-filter on L which is strictly finer than $\mathfrak{U}$.

Using the Zorn lemma, we deduce that

Proposition 14. *If $\mathfrak{F}$ is any S-filter on a locale L, there is an S-ultrafilter finer than $\mathfrak{F}$ on L.*

Proposition 15. *Let $\mathfrak{U}$ be an S-ultrafilter on a locale L, and let $k \in L$. Let S, T be sieves on k such that $S \cup T \in \mathfrak{U}(k)$ then either $S \in \mathfrak{U}(k)$ or $T \in \mathfrak{U}(k)$.*

Proof. If the affirmation is false, there exist sieves S, T on k that do not belong to $\mathfrak{U}(k)$, but $S \cup T \in \mathfrak{U}(k)$. Consider a function $\mathfrak{T} : L \to Sets$ defined by $\mathfrak{T}(k) = \{R \in \ Sieve(k) \mid R \cup S \in \mathfrak{U}(k)\}$.

Let us verify that $\mathfrak{T}$ is a S-filter on a L: in fact, for any object k of L, we have

- (F_1) if $R^{'} \in \mathfrak{T}(k)$ then $R^{'} \cup S \in \mathfrak{U}(k)$; and if $R^{''}$ is a sieve on k such that $R' \subseteq R''$, then $R^{''} \cup S \in \mathfrak{U}(k)$. Consequently $R^{''} \in \mathfrak{T}(k)$.
- (F_2) We must show that every finite intersection of sieves of $\mathfrak{T}(k)$ belongs to $\mathfrak{T}(k)$; indeed, let $(R_i)_{i=1,\cdots,n}$ be a finite collection of sieves on k such that $R_i \cup S \in \mathfrak{U}(k)$, for all $i = 1...n$, then $(R_1 \cup S) \cap (R_2 \cup S) \cap \cdots \cap (R_n \cup S) = (\bigcap_{i=1}^{n} R_i) \cup S \in \mathfrak{U}(k)$. which is equivalent to saying that $(\bigcap_{i=1}^{n} R_i) \in \mathfrak{T}(k)$.
- (F_3) If $R^{'} \in \mathfrak{T}(k)$ then $R^{'} \cup S \in \mathfrak{U}(k)$; and $h^*(R^{'} \cup S) \in \mathfrak{T}(n)$ for any arrow $h : n \to k$; in other words, $h^*(R^{'}) \cup h^*(S) \in \mathfrak{T}(n)$, therefore $h^*(R^{'}) \in \mathfrak{T}(n)$.
- (F_4) Evidently, the empty sieve is not in $\mathfrak{T}(k)$.

Therefore $\mathfrak{T}$ is a S-filter finer than $\mathfrak{U}$, since $T \in \mathfrak{T}(k)$; but this contradicts the hypothesis than $\mathfrak{U}$ is an S-ultrafilter. □

Corollary 5. *If the union of a finite sequence $(S_i)_{i=1,\cdots,n}$ of sieves on k belongs to the image, $\mathfrak{U}(k)$, of an object k under an ultraS-filter $\mathfrak{U}$, then at least one of the S_i belongs to $\mathfrak{U}(k)$.*

Proof. The proof is a simple use of induction on n. □

8.4. S-filters and Grothendieck topologies on locales

Definition 21.

1. If J_1 and J_2 are Grothendieck topologies on a locale L, we say that $J_1 \preceq J_2$ if and only if $J_1(k) \subseteq J_2(k)$ for all $k \in L$.
2. In the same way, if $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are S-filters on L, we say that $\mathfrak{F}_1 \preceq \mathfrak{F}_2$ if and only if $\mathfrak{F}_1(k) \subseteq \mathfrak{F}_2(k)$ for all objects $k \in L$.

It is easy to verify that this definition produces two order relations on Grothendieck topologies and S-filters respectively.

In this way, we have the following facts:

Lemma 4. *Every S-filter on a locale is a Grothendieck topology on the same locale.*

Lemma 5. *Let $\mathfrak{F}$ be a S-filter on a locale L and let J be a Grothendieck topology on the same locale. If $J \preceq \mathfrak{F}$ then J is a S-filter.*

Proof.

- (F_1) Given any $k \in L$, it is clear that if S is a sieve in $J(k)$ and R is a sieve on k such that $S \subseteq R$, then $R \in J(k)$;
- (F_2) let $(R_i)_{i=1,\cdots,n}$ be a finite collection of sieves on $J(k)$, then $\bigcap_{i=1}^{n} R_i \in J(k)$, and therefore $\bigcap_{i=1}^{n} R_i \in \mathfrak{F}(k)$ (and consequently is not empty);
- (F_3) If $S \in J(k)$, and $m \leqslant k$ then certainly $S \cap (\downarrow m)$ is in $\mathfrak{F}(m)$;
- (F_4) the empty sieve is neither in $\mathfrak{F}(k)$ nor in $J(k)$.

□

8.5. Systems of Neighborhoods on Locales

The basic ideas used in in this section can be found in P.T. Jhonstone book "Stone Spaces" ([7] pages 41-42) or in the S. MacLane and I. Moerdijk book "Sheaves in Geometry and Logic" (see [4] pages 470-473): A point of a topological space X is the same thing as a continuous map $1 \to X$, where 1 is the one-point space, it seems reasonable to define a point of a locale L to be a continuous map $\Omega(1) = 2 \to L$, i.e. a frame homomorphism $p : L \to 2$. such a map is completely determined by its kernel $p^{-1}(0)$ or its dual kernel $p^{-1}(1)$, which are respectively a prime ideal and a prime filter of L.

In what follows, we are going to use the frame homomorphisms $p : L \to 2$ as points of the locale L.

Definition 22. Let (L, J) be a locale equipped with a Grothendieck topology, let $k \in L$ and let p be a point of L such that $k \in p^{-1}(0)$. A sieve V in $J(k)$, is said to be a $\mathfrak{G}$-neighborhood of p if $V \subseteq p^{-1}(0)$.

Definition 23. Let (L, J) be a locale equipped with a Grothendieck topology. A cover-neighborhood of (L, J) is a function $\mathcal{N}$ which assigns to each object $(k, J(k))$ of (L, J) and to each point p of L, for which $k \in p^{-1}(0)$, a collection $\mathcal{N}_p(k)$ of sieves of k such that each sieve in $\mathcal{N}_p(k)$ contains a $\mathfrak{G}$-neighborhood of p.

Proposition 16. *Let L be a locale, and let $k \in L$. Then the pair $(k, \mathcal{N}_p(k))$, where $\mathcal{N}_p(k)$ is the collection of all cover-neighborhoods of a point p (for which $k \in p^{-1}(0)$) is an object of L equipped with cover-neighborhood.*

Proof.

(i) If $S \in \mathcal{N}_p(k)$ and R is a sieve on k such that $S \subseteq R$, then $R \in \mathcal{N}_p(k)$, because there is a $\mathfrak{G}$-neighborhood V of p_C such that $V \subseteq S \subseteq R$;

(ii) let $\{S_1, S_2, \cdots, S_n\}$ be a finite collection of sieves of $\mathcal{N}_p(k)$, then there exists a collection $\{V_1, V_2, \cdots, V_n\}$ of $\mathfrak{G}$-neighborhood of p_C such that $V_i \subseteq S_i$ for $I = 1, 2, \cdots n$, therefore $\bigcap_{i=1}^n V_n \subseteq \bigcap_{i=1}^n S_n$ and $\bigcap_{i=1}^n S_n \in \mathcal{N}_p(Ck)$;

(iii) the empty sieve is not in $\mathcal{N}_p(k)$ (each sieve contains a point).

□

In this case, we say that the point p of l is a **limit point** of $\mathcal{N}_p(k)$.

Definition 24. Let (L, J) be a locale equipped with a Grothendieck topology; let $\mathfrak{F}$ be a S-filter on L and let $k \in L$.

1. We shall say that $\mathfrak{F}(k)$ **converges** to a point p of L, for which $k \in p^{-1}(0)$, if $\mathcal{N}_p(k) \subseteq \mathfrak{F}(k)$.
2. The closure of a sieve A on k is the collection of all points p of L (satisfying $k \in p^{-1}(0)$) such that every cover-neighborhood of p meets A.
3. A point p is a cluster point of $\mathfrak{B}(k)$ -the image under the S-filter base $\mathfrak{B}$ of k- if it lies in the closure of all the sieves on $\mathfrak{B}(k)$.
4. A point p of C is a cluster point of $\mathfrak{F}(C)$ -the image under the S-filter $\mathfrak{F}$ of C- if it lies in the closure of all the sieves on $\mathfrak{F}(C)$.

Proposition 17. *Let (L, J) be a locale equipped with a Grothendieck topology; let $\mathfrak{F}$ be a S-filter on L and let $k \in L$. The point p of L is a cluster point of $\mathfrak{F}(k)$ if and only if there exists a S-filter $\mathcal{G}$ finer than $\mathfrak{F}$ such that $\mathcal{G}(k)$ **converges** to p.*

Proof. Let us begin by assuming that the point p of L is a cluster point of $\mathfrak{F}(k)$; from Definition 9, it follows that for each sieve A in $\mathfrak{F}(k)$, every $\mathfrak{G}$-neighborhood V of p meets A. We need to show that the collection $\mathcal{B}(k) = \{A \cap V \mid V \text{ is a } \mathfrak{G}\text{-neighborhood of } p\}$ define a base for an S-filter $\mathcal{G}$ finer than $\mathfrak{F}$.in such a way that $\mathcal{G}(C)$ **converges** to p.

Indeed,

(B_1) Let $A \cap V$, $A \cup W$ two elements of the collection $\mathcal{B}(k)$, since $(A \cap V) \cap (A \cup W) = A \cup (V \cap W)$ and $V \cap W$ is a $\mathfrak{G}$-neighborhood of p, there exists U, a $\mathfrak{G}$-neighborhood of p such that $U \subseteq V \cap W$, and clearly $A \cap U \in \mathcal{B}(k)$;

(B_2) Obviously $\mathcal{B}(k)$ is not empty, and the empty sieve is not in $\mathcal{B}(C)$.

Now, if $\mathcal{G}$ is the S-filter generated by $\mathcal{B}$ then $\mathcal{G}$ is finer than $\mathfrak{F}$, and $\mathcal{G}(C)$ naturally converges to p.

Conversely, if there is a S-filter $\mathcal{G}$ finer than $\mathfrak{F}$ such that $\mathcal{G}(k)$ **converges** to p then each sieve R in $\mathfrak{F}(k)$ and each $\mathfrak{G}$-neighborhood U of p belongs to $\mathcal{G}$ and hence meet, so the point p of L is a cluster point of $\mathfrak{F}(C)$. □

Proposition 18. *Let (L, J) be a locale equipped with a Grothendieck topology; let $k \in L$ and let A be a sieve on k. The point p of L lies in the closure of A if and only if there is a S-filter $\mathcal{G}$ such that $A \in \mathcal{G}(k)$ and $\mathcal{G}(k)$ **converges** to p.*

Proof. Let us begin by assuming that The point p of L lies in the closure of A; from Definition 9, it follows that every $\mathfrak{G}$-neighborhood V of p meets A. Then $\mathcal{B}(k) = \{A \cap V \mid V \text{ is a } \mathfrak{G}\text{-neighborhood of } p\}$ is a base for a S-filter $\mathcal{G}$, in such a way that $\mathcal{G}(k)$ **converges** to p.

Conversely, if $A \in \mathcal{G}(k)$ and $\mathcal{G}(C)$ **converges** to p then p is a cluster point of $\mathcal{G}(k)$ and hence p lies in the closure of A. □

8.6. Compactness on Locales

In this section we consider a concept of compactness on locales entirely different from the one described by P. J Johnstone in [5].

Definition 25. . We shall say that an object $k \in L$

- Is **quasi-compact** if, for every S-filter $\mathfrak{F}$ on L, $\mathfrak{F}(k)$ has at least one cluster point.
- Is **Hausdorff** if, for every S-filter $\mathfrak{F}$ on L, $\mathfrak{F}(k)$ has no more that one limit point.
- Is **compact** if, for every S-filter $\mathfrak{F}$ on L, $\mathfrak{F}(k)$ is quasi-compact and Hausdorff.

Lemma 6. *Let (L, J) be a locale equipped with a Grothendieck topology. An object $k \in L$ is compact if and only if, for every S-ultrafilter $\mathfrak{U}$ on L, $\mathfrak{U}(k)$ is convergent.*

Proof. First suppose that $\mathfrak{F}$ is an S-filter on L. Proposition 14 ensures that, for every S-filter, there exists an S-ultrafilter $\mathfrak{U}$ finer than $\mathfrak{F}$, such that $\mathfrak{U}(k)$ converges to a point p on L, therefore p is a cluster point of $\mathfrak{F}(k)$.

Conversely, if, for an S-ultrafilter $\mathfrak{U}$, $\mathfrak{U}(k)$ has a cluster point then it converges to this point. $\square$

Proposition 19. *(Tychonoff) Let (L, J) be a locale equipped with a Grothendieck topology. Every meet (greatest lower bound) of compact objects in the category L is compact.*

Proof. Suppose we have chosen a collection $(k_i)_{i\in I}$ of compact objects in L; equivalently, for every family $(\mathfrak{U}_i)_{i\in I}$ of ultrafilters on L, $\mathfrak{U}_i(k_i)$ is convergent. Then the function $\mathfrak{B}$ which assigns to each object $k = \bigwedge_{i\in I} k_i$, the collection of sieves $\mathfrak{B}(k) = \left\{ \bigcap\limits_{i\in I} S_i \mid S_i \in \mathfrak{F}_i(k) \right\}$, where $S_i = t_{k_i}$ is the maximal sieve on k_i except for a finite number of indices, is basis of an ultrafilter on L. $\square$

Example 6. Let $L_{\mathbb{R}} = \Omega(\mathbb{R})$ be the locale of open subsets of the usual topolgy on the set $\mathbb{R}$ of real numbers. Let k_1 be the open interval $(-1, 1) \subset \mathbb{R}$, and let $p : \Omega(\mathbb{R}) \to 2$ be a point of $\Omega(\mathbb{R})$ such that $p^{-1}(0)$ is the ideal $!(-2, 2)$.

1. Let J be a Grothendieck topology on $L_{\mathbb{R}}$ defined by
$$J(k) = \begin{cases} \{!k_1, \{(\frac{-1}{r}\ \frac{1}{r}) \mid r \in \mathbb{Z}^+\}\} & \text{if } k = k_1 \\ \left\{!k, \{!k_1, \{(\frac{-1}{r}\ \frac{1}{r}) \mid r \in \mathbb{Z}^+\}\} \cap (!k)\right\} & \text{if } k \subseteq k_1 \\ \left\{!k, \{(\frac{-1}{r}\ \frac{1}{r}) \mid r \in \mathbb{Z}^+\} \cup \{\alpha, \beta)\}\right. \text{ if } k_1 \subseteq (\alpha, \beta) \subseteq k & \\ \{!k\} & \text{if } k_1 \nsubseteq k. \end{cases}$$
2. Let $\mathcal{N}$ be a cover-neighborhood of $(L_{\mathbb{R}}, J)$ and let $\mathfrak{F}$ be S-filter on $L_{\mathbb{R}}$ so that $\mathcal{N}_p(k_1) = J(k_1) \subseteq \mathfrak{F}(k_1)$.
3. We may conclude from Propositions 16, 17 and 18 that $k_1 = (-1, 1) \subset \mathbb{R}$ is a compact object of $\Omega(\mathbb{R})$.

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