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N-Coincidence and Common N-fixed Point for Hybrid Pair of Mappings in Partially Ordered Metric Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Author RAR designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Author IMS managed the analysis of the study and literature searches. Both authors read and approved the final manuscript.

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ABSTRACT

In present paper we introduce the concepts of mixed g-monotone property and Δ_g-symmetric property, where g is single valued mapping, for multi-valued mapping under any number of variables and use it to obtain some existence and uniqueness fixed points for hybrid pair of mappings under general contractive conditions in partially ordered metric spaces. Our results are generalizations of several results in this direction. We equipped this paper with examples in order to illustrate the effectiveness of our generalizations.

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1 INTRODUCTION

The study of fixed points for multi-valued contraction mappings using the Hausdorff metric was initiated by Nadler [1] and Markin [2]. Later many authors developed the existence of fixed points for various multi-valued contractive mappings under different conditions. For details, we refer the reader to [3, 4, 5, 6, 7, 8, 9] and the references therein. The theory of multi-valued mappings has applications in control theory, convex optimization, differential inclusion and economics.

Let (X,d) be a metric space. We denote by $CB(X)$ the family of all nonempty closed and bounded subsets of X and $CL(X)$ the set of all nonempty closed subsets of X.

For $A,B \in CB(X)$ and $x \in X$, we denote

$$D(x,A) = \inf_{a\in A} d(x,a)$$

Let H be the Hausdorff metric in $CB(X)$ induced by the metric d on X, that is

$$H(A,B) = \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(A,b)\}.$$

For any $A,B \in CB(X)$ and $a \in A$, we have

$$D(a,B) \le \sup_{a\in A} D(a,B) \le H(A,B).$$

Remark 1.1. [1] Let $A,B \in CB(X)$ and $a \in A$. If $\eta > 0$, then it is a simple consequence of the definition of $H(A,B)$ that there exists $b \in B$ such that $d(a,b) \le H(A,B) + \eta$.

Lemma 1.1. *[1] Let $A,B \in CB(X)$ and $\alpha > 1$. Then for every $a \in A$, there exists $b \in B$ such that $d(a,b) \le \alpha H(A,B)$.*

Definition 1.1. [1] An element $x \in X$ is said to be a fixed point of a set valued mapping $T : X \to CB(X)$ if and only if $x \in Tx$.

In 1969, Nadler [1] extended the famous Banach contraction principle from single-valued mapping to multi-valued mapping and prove the following theorem.

Theorem 1.2. *[1] Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0,1]$ such that*

$$H\big(Tx,Ty\big) \le cd(x,y),$$

for all $x,y \in X$. Then T has a fixed point.

In [10] Bhaskar and Lakshmikantham proved the existence of coupled fixed point for a single valued mapping $F : X \times X \to X$ under weak contractive conditions and as an application they proved the existence of a unique solution of a boundary value problem associated with a first order ordinary differential equation. Then, Lakshmikantham and Ćirić [11] obtained a coupled coincidence and coupled common fixed point of two single valued mappings $F : X\times X \to X$ and $g : X \to X$ in the frame work of ordered complete metric space.

The concept of coupled fixed point for multi valued mapping $F : X \times X \to CB(X)$ was introduced by Beg and Butt [12] who followed the technique of Bhaskar and Lakshmikantham to define the mixed monotone property for F and give sufficient conditions for the existence of its coupled fixed point (not necessarily unique) in an ordered space $(X,d,\preceq)$. On the other hand, Samet and Vetro [13] established two coupled fixed point theorems for multi valued nonlinear contraction mapping $F : X \times X \to CL(X)$ with Δ-symmetric property in partially ordered metric spaces. Later several authors proved coupled (tripled) coincidence and common fixed point theorems for hybrid pairs in partially ordered metric spaces and other spaces, we refer to [14, 15, 16, 17, 18, 19, 20].

Definition 1.2. [15] Let X be a nonempty set, $F : X \times X \to 2^X$ (collection of all nonempty subsets of X) and $g : X \to X$. An element $(x,y) \in X \times X$ is called

(i) coupled fixed point of F if $x \in F(x,y)$ and $y \in F(y,x)$

(ii) coupled coincidence point of a hybrid pair F, g if $g(x) \in F(x,y)$ and $g(y) \in F(y,x)$

(iii) common coupled fixed point of a hybrid pair F, g if $x = g(x) \in F(x, y)$ and $y = g(y) \in F(y, x)$.

We denote the set of coupled coincidence point of mappings F and g by $C(F, g)$. Note that if $(x, y) \in C(F, g)$, then (y, x) is also in $C(F, g)$.

Definition 1.3. [15] Let $F : X \times X \to 2^X$ be a multi valued mapping and g be a self mapping on X. The hybrid pair F, g is called w-compatible if $g(F(x, y)) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.

The following is the main results of Beg and Butt [12].

Definition 1.4. Let X be a partially ordered set and $F : X \times X \to CB(X)$ be a set valued mapping. F is said to be a mixed monotone mapping if F is order-preserving in x and order-reversing in y i.e., $x_1 \preceq x_2$, $y_2 \preceq y_1$, $x_i, y_i \in X (i = 1, 2)$ implies for all $u_1 \in F(x_1, y_1)$ there exists $u_2 \in F(x_2, y_2)$ such that $u_1 \preceq u_2$ and for all $v_1 \in F(y_1, x_1)$ there exists $v_2 \in F(y_2, x_2)$ such that $v_2 \preceq v_1$.

If $\preceq$ is the relation defined on the set X we can define the partial order on the product space $X \times X$ as

$$(u, v) \preceq (x, y) \Leftrightarrow u \preceq x \ \texttt{and}$$
$$v \succeq y \quad \forall (u, v), (x, y) \in X \times X.$$

The product metric on $X \times X$ is defined as

$$d((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2),$$
$$\texttt{for all} \ x_i, y_i \in X \ \texttt{and} \ (i = 1, 2).$$

Theorem 1.3. *[12] Let $(X, d, \preceq)$ be a partially ordered complete metric space and $F : X \times X \to CB(X)$ be a set valued mapping with non empty closed bounded values satisfying:*

(1) *There exists $\kappa \in (0, 1)$ with*

$$H(F(x, y), F(u, v)) \leq \frac{\kappa}{2} d\big((x, y), (u, v)\big),$$

for all $(x, y) \succeq (u, v)$.

(2) *Given $x_i, y_i \in X$, $(i = 1, 2)$ with $x_1 \preceq x_2$ and $y_2 \preceq y_1$ then for all $u_1 \in F(x_1, y_1)$ there exists $u_2 \in F(x_2, y_2)$ with $u_1 \preceq u_2$ and for all $v_1 \in F(y_1, x_1)$ there exists $v_2 \in F(y_2, x_2)$ with $v_2 \preceq v_1$ provided $d\big((u_1, v_1), (u_2, v_2)\big) < 1$.*

(3) *There exists $x_0, y_0 \in X$ and some $x_1 \in F(x_0, y_0)$, $y_1 \in F(y_0, x_0)$ with $x_0 \preceq x_1$ and $y_0 \succeq y_1$ such that $d\big((x_0, y_0), (x_1, y_1)\big) < 1 - \kappa$.*

(4) *If a nondecreasing sequence $\{x_n\} \to x \in X$ then $x_n \preceq x$, for all n and if a nonincreasing sequence $y_n \to y \in X$ then $y_n \succeq y$, for all n.*

Then F has a coupled fixed point.

Roldán et al. [21] introduced the concepts of coincidence and common fixed points between two single valued mappings in any number of variable and gave the following definitions.

Definition 1.5. [21] Let $g : X \to X$ be mapping and $(X, d, \preceq)$ be an ordered metric space, then X is said to have the sequential g-monotone property if it verifies the following properties:

(i) If $\{x_m\}_{m \geq 0}$ is a non-decreasing sequence in X and $\lim_{m \to \infty} x_m = x$, then $gx_m \preceq gx$ for all $m \geq 0$,

(ii) If $\{y_m\}_{m \geq 0}$ is a non-increasing sequence in X and $\lim_{m \to \infty} y_m = y$, then $gy_m \succeq gy$ for all $m \geq 0$.

If g is the identity mapping, then X is said to have the sequential monotone property.

Definition 1.6. [21] We say that F and g are commuting if $gF(x_1, \ldots, x_n) = F(gx_1, \ldots, gx_n)$ for all $x_1, \ldots, x_n \in X$, and they are weakly compatible if they commute at their coincidence points.

Fix a partition $\{A, B\}$ of the set $\Lambda_n = \{1, 2, \ldots, n\}$, that is, $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$, we will denote

$$\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A \ \text{ and } \ \sigma(B) \subseteq B\}$$

and

$$\acute{\Omega}_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B \ \text{ and } \ \sigma(B) \subseteq A\}.$$

If $(X, \preceq)$ is a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, we will use the following notation

$$x \preceq_i y \Leftrightarrow \begin{cases} x \preceq y, & i \in A, \\ x \succeq y, & i \in B. \end{cases}$$

Definition 1.7. [21] Let $(X, \preceq)$ be a partially ordered set and $F : X^4 \to X$ be a mapping. We say that F has the mixed g-monotone property if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, that is, for all $x_1, x_2, \ldots, x_n, y, z, X$ and all i, i.e., for all $i \in \Lambda_n$.

$$gy \preceq gz \Rightarrow F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preceq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

Henceforth, let $\sigma_1, \ldots, \sigma_n, \tau : \Lambda_n \to \Lambda_n$ be $n+1$ mappings and let ϕ be the $(n+1)$-tuple $(\sigma_1, \ldots, \sigma_n, \tau)$.

Definition 1.8. [21] A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a ϕ-coincidence point of the mappings F and g if

$$F(x_{\sigma_i(1)}, \ldots, x_{\sigma_i(n)}) = gx_{\tau(i)} \text{ for all } i.$$

If g is the identity mapping on X, then $(x_1, \ldots, x_n) \in X^n$ is called a $\phi-$ fixed point of the mappings F.

Definition 1.9. A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a common ϕ-fixed point of the mappings F and g if

$$F(x_{\sigma_i(1)}, \ldots, x_{\sigma_i(n)}) = gx_{\tau(i)} = x_{\tau(i)} \text{ for all } i.$$

Theorem 1.4. *[21] Let $(X, d, \preceq)$ be a complete ordered metric space. Let $\phi = (\sigma_1, \sigma_2, \ldots, \sigma_n, \tau)$ be a $(n+1)$-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself such that $\tau \in \Omega_{A,B}$ is a permutation and verifying that $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property on X, $F(X^n) \subseteq g(X)$ and g commutes with F. Assume that there exists $k \in [0, 1)$ verifying*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq k \max_{1 \leq i \leq n} d(gx_i, gy_i)$$

for which $gx_i \preceq_i gy_i$ for all i. Suppose either F is continuous or X has the sequential g-monotone property. If there exist $x_0^1, \ldots . x_0^n \in X$ verifying

$$gx_0^{\tau(i)} \preceq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}) \text{ for all } i.$$

Then F and g have, at least, one ϕ-coincidence point.

By using Δ-symmetric property Samet and Vetro [13] established two coupled fixed point theorems for set valued mapping $F : X \times X \to CL(X)$ by considering metric spaces endowed with partial order as a generalization and extension of the recent result of Ćirić [4].

Let (X, d) be a metric space endowed with a partial order. We recall some of results that we use it in second section.

Definition 1.10. [13] A function $f : X \times X \to R$ is called lower semi-continuous if and only if for any $\{x_n\} \subset X$, $\{y_n\} \subset X$ and $(x, y) \in X \times X$, we have

$$\lim_{n \to \infty} (x_n, y_n) = (x, y) \Rightarrow f(x, y) \leq \liminf_{n \to \infty} f(x_n, y_n).$$

Let $G : X \to X$ be a given mapping. We define the set $\Delta \subseteq X \times X$ by $\Delta = \{(x, y) \in X \times X | G(x) \preceq G(y)\}$.

Definition 1.11. [13] Let $F : X \times X \to CL(X)$ be a given mapping. We say that F is a Δ-symmetric mapping if and only if

$$(x, y) \in \Delta \Rightarrow F(x, y) \times F(y, x) \subseteq \Delta.$$

Definition 1.12. [13] Let $F : X \times X \to CL(X)$ be a given mapping. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if and only if

$$x \in F(x, y) \;\; \texttt{and} \;\; y \in F(y, x).$$

Theorem 1.5. *[13] Let (X, d) be a complete metric space endowed with a partial order $\preceq$. We assume that $\Delta \neq \emptyset$, i.e., there exists $(x_0, y_0) \in \Delta$. Let $F : X \times X \to CL(X)$ be a Δ-symmetric mapping. Suppose that the function $f : X \times X \to [0, \infty)$ defined by*

$$f(x, y) = D(x, F(x, y)) + D(y, F(y, x)) \;\; \textit{for all} \;\; x, y \in X,$$

is lower semi-continuous and that there exists a function $\phi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying

$$\limsup_{r \to t^+} \phi(r) < 1 \quad \textit{for each} \;\; t \in [0, \infty).$$

Assume that for any $(x, y) \in \Delta$ there exist $u \in F(x, y)$ and $v \in F(y, x)$ satisfying

$$\sqrt{\phi(f(x, y))}\big[d(x, u) + d(y, v)\big] \leq f(x, y)$$

such that

$$f(u, v) \leq \phi\big(f(x, y)\big)\big[d(x, u) + d(y, v)\big].$$

Then F admits a coupled fixed point, i.e., there exists $z = (z_1, z_2) \in X \times X$ such that $z_1 \in F(z_1, z_2)$ and $z_2 \in F(z_2, z_1)$.

The main results of this paper are presented in sections two and three. Section 2 is devoted to prove a N-coincidence point theorems for hybrid pair of mappings in partially ordered metric space via mixed monotone property without appeal to the completeness or closeness of the underlying space or the continuity of the mappings involved therein. Section 3 is devoted to prove N-coincidence point for hybrid pair of nonlinear contractions in partially ordered metric space by using Δ_g-symmetric property instead of mixed g-monotone property. Examples are given to support our results.

2 N-COINCIDENCE POINTS FOR HYBRID PAIR OF MAPPINGS VIA MIXED MONOTONE PROPERTY

Now, we define mixed g-monotone property for multi-valued mapping F with n-variable and apply this to obtain the existence of N-coincidence point of these mappings.

Definition 2.1. Consider $A, B \subseteq X$, we can define the following relations on the power set of X

- $A \preceq^1 B$ if for any $a \in A$ we can find $b \in B$ such that $a \preceq b$,
- $A \preceq^2 B$ if for any $b \in B$ we can find $a \in A$ such that $a \preceq b$,
- $A \preceq^3 B$ if $A \preceq^1 B$ and $A \preceq^2 B$.

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set, $F : X^n \to CL(X)$ be multi-valued mapping with n-variable and $g : X \to X$ be single valued mapping. We say that F has the mixed g-monotone property if $F(x, y, z, w)$ is g-monotone non-decreasing in argument of A and g-monotone non-increasing in argument of B, that is, for any $x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n, y$ and $z \in X$ we have

$$\begin{aligned} gy \preceq gz \quad &\Rightarrow \quad F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preceq_i^{\tau(i)} F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) \\ &\Rightarrow \quad \begin{cases} F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preceq^{\tau(i)} F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n), & i \in A; \\ F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \succeq^{\tau(i)} F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n), & i \in B, \end{cases} \end{aligned}$$

where,

$$\tau(i) = \begin{cases} 1, & i \in A; \\ 2, & i \in B. \end{cases}$$

Theorem 2.1. *Let $(X, \preceq, d)$ be an ordered metric space and $\phi = (\sigma_1, \ldots, \sigma_n)$ be a (n)-tuple of mappings from $\Lambda_n = \{1, 2, \ldots, n\}$ into itself such that, $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega_{\acute{A},B}$ if $i \in B$. Let $F : X^n \to CL(X)$ and $g : X \to X$ be hybrid pair of mappings such that F has the mixed g-monotone property, $F(X^n) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Assume that there exist $a_i \in R, i \in \Lambda_n$ verifying $\sum_{i=1}^n a_i < 1$ and*

$$H\big(F(x^1, \ldots, x^n), F(y^1, \ldots, y^n)\big) \leq \sum_{j=1}^{n} a_j d(gx^j, gy^j) \tag{2.1}$$

for which $gx^i \preceq_i gy^i$. If there exist $x_0^1, \ldots x_0^n \in X$ such that

$$\{g(x_0^i)\} \preceq_i^{\tau(i)} F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)}), \text{ for all } i \in \Lambda_n \tag{2.2}$$

and X has the sequential g-monotone property. Then F and g have, at least, one N-coincidence point or ϕ-coincidence point as mentioned of Definition 1.8 of Roldán et al. [21].

Proof. By (4.2), $F(X^n) \subseteq g(X)$ and $F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)})$ is well defined and nonempty for all $i \in \Lambda_n$ and for any $(x_0^1, \ldots, x_0^n) \in X^n$, we can find $(x_1^1, \ldots, x_1^n) \in X^n$ such that

$$gx_1^i \in F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)}) \tag{2.3}$$

and

$$gx_0^i \preceq_i gx_1^i, \text{ for all } i \in \Lambda_n. \tag{2.4}$$

If $a_1 = \ldots = a_n = 0$ then

$$\begin{aligned} D\big(gx_1^i, F(x_1^{\sigma_i(1)}, \ldots, x_1^{\sigma_i(n)})\big) &\leq H\big(F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)}), F(x_1^{\sigma_i(1)}, \ldots, x_1^{\sigma_i(n)})\big) \\ &\leq \sum_{j=1}^{n} a_j d(gx_0^{\sigma_i(j)}, gx_1^{\sigma_i(j)}) = 0 \\ &\Rightarrow gx_1^i \in \overline{F(x_1^{\sigma_i(1)}, \ldots, x_1^{\sigma_i(n)})} = F(x_1^{\sigma_i(1)}, \ldots, x_1^{\sigma_i(n)}). \end{aligned}$$

Hence $(x_1^1, \ldots, x_1^n)$ is N-coincidence point for F and g. Now assume that $a_i > 0$ for some $i \in \Lambda_n$. For the point $(x_1^1, \ldots, x_1^n)$, we can find another point $(x_2^1, \ldots, x_2^n) \in X^n$ such that

$$gx_2^i \in F(x_1^{\sigma_i(1)}, \ldots, x_1^{\sigma_i(n)}), \text{ for all } i \in \Lambda_n.$$

Continuing this process we can construct sequences $\{x_m^1\}_{m \geq 0}, \ldots, \{x_m^n\}_{m \geq 0}$ such that

$$gx_{m+1}^i \in F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(n)}), \text{ for all } m \geq 0 \text{ and } i \in \Lambda_n.$$

By induction methodology for $m \geq 0$, we shall prove that

$$gx_m^i \preceq_i gx_{m+1}^i, \text{ for all } i \in \Lambda_n. \tag{2.5}$$

Indeed, from equations (2.3) and (2.4), we have

$$gx_0^i \preceq_i gx_1^i \in F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)}),$$

Suppose that (2.5) is true for some $m \geq 0$ and we are going to prove it for $m+1$. Now we have to distinguish between wether $i \in A$ or $i \in B$,

(Case 1) Suppose that $i \in A$ $(\sigma_i \in \Omega_{A,B})$.
$gx_{m+1}^i \in F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$ for this argument $x_m^{\sigma_i(j)}$, we have two subcases,

(I) If $j \in A$ (where F is g-monotone non-decreasing), $\sigma_i(j) \in A$ (i.e., $gx_m^{\sigma_i(j)} \preceq gx_{m+1}^{\sigma_i(j)}$). Thus, $F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \preceq^{\tau(i)=1} F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$.

(II) $j \in B$ (where F is g-monotone non-increasing), $\sigma_i(j) \in B$ (i.e., $gx_m^{\sigma_i(j)} \succeq gx_{m+1}^{\sigma_i(j)}$). Thus, $F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \preceq^{1} F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$.
That is,

$$F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \preceq^{1} F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \ \ \forall \ \ i \in A, j \in \Lambda_n$$

and

$$\begin{aligned} F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) &\preceq^{1} F(x_{m+1}^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \\ &\vdots \\ &\preceq^{1} F(x_{m+1}^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \\ &\vdots \\ &\preceq^{1} F(x_{m+1}^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_{m+1}^{\sigma_i(n)}). \end{aligned}$$

Therefore, for $gx_{m+1}^i \in F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(n)})$ there exist $gx_{m+2}^i \in F(x_{m+1}^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(n)})$

$$gx_{m+1}^i \preceq gx_{m+2}^i, \ \ i \in A. \tag{2.6}$$

(Case 2) If $i \in B$ $(\sigma_i \in \Omega'_{A,B})$.
$gx_{m+1}^{\tau(i)} = F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$ for this argument $x_m^{\sigma_i(j)}$, we have two subcases,

(I) If $j \in A$ (Place where F is g-monotone non-decreasing), $\sigma_i(j) \in B$ (i.e., $gx_m^{\sigma_i(j)} \succeq gx_{m+1}^{\sigma_i(j)}$). Thus, $F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \succeq^{\tau(i)=2} F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$.

(II) $j \in B$ (where F is g-monotone non-increasing), $\sigma_i(j) \in A$ (i.e., $gx_m^{\sigma_i(j)} \preceq gx_{m+1}^{\sigma_i(j)}$). Thus, $F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \succeq^{2} F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)})$.
We conclude that

$$F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \succeq^{2} F(x_m^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \ \ \forall \ \ i \in B, j \in \Lambda_n$$

and

$$F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(j)}, \ldots, x_m^{\sigma_i(n)}) \succeq^{2} F(x_{m+1}^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(j)}, \ldots, x_{m+1}^{\sigma_i(n)}).$$

Therefore, for $gx_{m+1}^i \in F(x_m^{\sigma_i(1)}, \ldots, x_m^{\sigma_i(n)})$ there exist $gx_{m+2}^i \in F(x_{m+1}^{\sigma_i(1)}, \ldots, x_{m+1}^{\sigma_i(n)})$ such that

$$gx_{m+1}^i \succeq gx_{m+2}^i, \ \ i \in B. \tag{2.7}$$

From inequalities (2.6) and (2.7), we get

$$gx^i_{m+1} \preceq_i gx^i_{m+2}.$$

Thus, inequality (2.5) is true for any $i \in \Lambda_n$ and $m \geq 0$. Note that the inequality (2.5) implies ($gx^{\sigma_i(j)}_{m-1} \preceq_j gx^{\sigma_i(j)}_m$, if $i \in A$, or $gx^{\sigma_i(j)}_{m-1} \succeq_j gx^{\sigma_i(j)}_m$, if $i \in B$).

Then we use the previous fact, the contraction condition (4.1) and Remark 1.1 to assert that the sequences $\{gx^i_m\}_{m\geq 0}$ are Cauchy for all $i \in \Lambda_n$ as follows:

$$d(gx^i_m, gx^i_{m+1}) \leq H\big(F(x^{\sigma_i(1)}_{m-1}, \ldots, x^{\sigma_i(n)}_{m-1}), F(x^{\sigma_i(1)}_m, \ldots, x^{\sigma_i(n)}_m)\big) + h^m \leq \sum_{j=1}^n a_j d(gx^{\sigma_i(j)}_{m-1}, gx^{\sigma_i(j)}_m) + h^m$$

$$\leq \sum_{j=1}^n a_j \max_{1\leq i\leq n} d(gx^i_{m-1}, gx^i_m) + h^m, \ \ \texttt{for all } i \texttt{ and } h < 1.$$

Consider $\max_{1\leq i\leq n} d(gx^i_m, gx^i_{m+1}) = d(gx^k_m, gx^k_{m+1}) = \delta_m$ for some $k \in \Lambda_n$ and since the above inequality hold for any element in Λ_n then we have,

$$\begin{aligned}
d(gx^k_m, gx^k_{m+1}) &\leq \sum_{j=1}^n a_j \max_{1\leq i\leq n} d(gx^i_{m-1}, gx^i_m) + h^m \\
\delta_m &\leq \sum_{j=1}^n a_j \delta_{m-1} + h^m \\
&\leq \lambda\delta_{m-1} + h^m, \ \ \lambda = \sum_{j=1}^n a_j < 1 \\
&\leq \lambda(\lambda\delta_{m-2} + h^{m-1}) + h^m \\
&\leq \lambda^2(\lambda\delta_{m-3} + h^{m-2}) + \lambda h^{m-1} + h^m \\
&\leq \lambda^3\delta_{m-3} + \lambda^2 h^{m-2} + \lambda h^{m-1} + h^m \\
&\vdots \\
&\leq \lambda^m\delta_0 + (\lambda^{m-1}h + \lambda^{m-2}h^2 + \cdots + \lambda h^{m-1} + h^m) \\
&\leq \lambda^m\delta_0 + m\eta^m, \eta = \max\{\lambda, h\} < 1 \\
\Rightarrow d(gx^i_m, gx^i_{m+1}) &\leq \delta_m \leq \lambda^m\delta_0 + m\eta^m. \qquad (2.8)
\end{aligned}$$

For a fixed i we use the triangle inequality and (2.8) to obtain:

$$\begin{aligned}
d(gx^i_m, gx^i_{m+p}) &\leq d(gx^i_m, gx^i_{m+1}) + d(gx^i_{m+1}, gx^i_{m+2}) + \cdots + d(gx^i_{m+p-1}, gx^i_{m+p}) \\
&\leq \big(\lambda^m + \lambda^{m+1} + \cdots + \lambda^{m+p-1}\big)\delta_0 + \big(m\eta^m + (m+1)\eta^{m+1} + \cdots + (m+p-1)\eta^{m+p-1}\big) \\
&\leq \lambda^m\big(1 + \lambda + \cdots + \lambda^{p-1}\big)\delta_0 + \sum_{i=m}^{m+p-1} i\eta^i \\
&\leq \lambda^m \frac{1-\lambda^p}{1-\lambda}\delta_0 + \sum_{i=m}^{m+p-1} i\eta^i \ \ \to 0 \ \text{ as } \ m \to \infty.
\end{aligned}$$

Therefore, $\{gx^i_m\}_{m\geq 0}$ are Cauchy sequences (for all $i \in \Lambda_n$) in $g(X)$. By the completeness of $g(X)$, there exist $\{gx^1, \ldots, gx^n\} \in g(X)$, such that

$$gx^i_m \to gx^i, \ \text{ as } \ n \to \infty \ \text{ for all } \ i \in \Lambda_n. \qquad (2.9)$$

Finally, we claim that the point $(x^1, \dots, x^n)$ is N-coincidence point of F and g. Suppose that X has the sequential g-monotone property, by (2.5) and (2.9) we have $gx^i_m \preceq_i gx^i_{m+1}$ and $gx^i_m \to gx^i$, as $m \to \infty$ for all $i \in \Lambda_n$, implying $gx^i_m \preceq_i gx^i$ and

$$\Big(gx_m^{\sigma_i(j)} \preceq_j gx^{\sigma_i(j)} \text{ or } gx_m^{\sigma_i(j)} \succeq_j gx^{\sigma_i(j)}\Big).$$

Now consider

$$\begin{aligned} D\big(F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}), gx^i\big) &\le H\big(F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}), F(x_m^{\sigma_i(1)}, \dots, x_m^{\sigma_i(n)})\big) + d(gx^i_{m+1}, gx^i) \\ &\le \sum_{j=1}^{n} a_j d(gx_m^{\sigma_i(j)}, gx^{\sigma_i(j)}) + d(gx^i_{m+1}, gx^i). \end{aligned} \tag{2.10}$$

This can be done because,

$$\begin{aligned} D\big(F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}), gx^i\big) &= \inf_{\xi \in F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)})} d(\xi, gx^i) \\ &\le \inf_{\xi \in F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)})} \big\{d(\xi, gx^i_{m+1}) + d(gx^i_{m+1}, gx^i)\big\} \\ &= \inf_{\xi \in F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)})} d(\xi, gx^i_{m+1}) + d(gx^i_{m+1}, gx^i) \\ &= D\big(F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}), gx^i_{m+1}\big) + d(gx^i_{m+1}, gx^i) \\ &\le H\big(F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}), F(x_m^{\sigma_i(1)}, \dots, x_m^{\sigma_i(n)})\big) + d(gx^i_{m+1}, gx^i) \end{aligned}$$

By (2.9), there exist $m_0, m_1, \dots, m_n \in N$ such that

$$d(gx^i_{m+1}, gx^i) < \frac{\epsilon}{2} \ \ \forall \ \ m \ge m_0, \quad for \ \epsilon > 0$$

and

$$d(gx_m^{\sigma_i(j)}, gx^{\sigma_i(j)}) < \frac{\epsilon}{2na_j} \ \ \forall \ \ m \ge m_j \ \ , j \in \Lambda_n.$$

Taking $m \ge \mu = \max\{m_0, m_1, \dots, m_n\}$ and using (2.10), we get

$$\begin{aligned} D\big(F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}), gx^i\big) &\le \Bigg(a_1 d(gx_m^{\sigma_i(1)}, gx^{\sigma_i(1)}) + \dots + a_n d(gx_m^{\sigma_i(n)}, gx^{\sigma_i(n)})\Bigg) + d(gx^i_{m+1}, gx^i) \\ &\le \Bigg(a_1 \frac{\epsilon}{2na_1} + \dots + a_n \frac{\epsilon}{2na_n}\Bigg) + \frac{\epsilon}{2} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ is an arbitrary, then $F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}) = gx^i$. That is, $(x^1, \dots, x^n)$ is a N-coincidence point of F and g. □

Example 2.2. *Let $X = [0, 1]$ with the Euclidean metric and the usual order contains a partially ordered complete metric space. Define $F : X^n \to CL(X)$ and $g : X \to X$ as*

$$F\big(x_1, \dots, x_n\big) = \left[0, \frac{\sin x_1 - \sin x_2 + \sin x_3 - \ldots + \sin x_n}{4n}\right]$$

and

$$g\big(x\big) = \frac{x}{2}.$$

Note that $\sin x_i < 1$*for any* $x_i \in [0,1]$*, then* $\left[\frac{\sin x_1 - \sin x_2 + \sin x_3 - \ldots + \sin x_n}{4n}\right] < \frac{\frac{n}{2}}{4n} < 1$*. Consider* $\sigma_1, \sigma_2, \ldots, \sigma_n : \Lambda_n \to \Lambda_n$ *defined as the following form*

$$
\begin{aligned}
\tau = \sigma_1 &= \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix} \\
\sigma_2 &= \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 2 & 1 & 2 & \ldots & n-1 \end{pmatrix} \\
&\vdots \\
\sigma_i &= \begin{pmatrix} 1 & 2 & 3 & \ldots & i & \ldots & n \\ i & i-1 & i-2 & \ldots & 1 & \ldots & n-i+1 \end{pmatrix} \\
&\vdots \\
\sigma_n &= \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ n & n-1 & n-2 & \ldots & 1 \end{pmatrix}.
\end{aligned}
$$

Let A *be the set of odd numbers and* B *be the set of even numbers in* Λ_n*.*

We have, $g(X) = X$ *is complete,* $\bigcup F\big(x_1, \ldots, x_n\big) \subseteq g(X)$ *and* X *has the sequential monotone property. Also, we have for all* $(x_1, \ldots, x_n), (u_1, \ldots, u_n) \in X^n$ *and* $gx_i \leq_i gu_i$

$$
\begin{aligned}
H\Big(F\big(x_1, \ldots, x_n\big), F\big(u_1, \ldots, u_n\big)\Big) &= \left|\frac{\sin x_1 - \sin x_2 + \ldots + \sin x_n}{4n} - \frac{\sin u_1 - \sin u_2 + \ldots + \sin u_n}{4n}\right| \\
&= \frac{1}{4n}\Big|\big(\sin x_1 - \sin u_1\big) + \big(\sin u_2 - \sin x_2\big) + \ldots + \big(\sin x_n - \sin u_n\big)\Big| \\
&\leq \frac{1}{4n}\Big[\big|\sin x_1 - \sin u_1\big| + \big|\sin x_2 - \sin u_2\big| + \ldots + \big|\sin x_n - \sin u_n\big|\Big] \\
&\leq \frac{1}{4n}\Big[\big|x_1 - u_1\big| + \ldots + \big|x_n - u_n\big|\Big] \\
&= \frac{1}{4n}\sum_{i=1}^{n}\big|x_i - u_i\big| \\
&= \frac{1}{2n}\sum_{i=1}^{n}\big|\frac{x_i}{2} - \frac{u_i}{2}\big| \\
&= \frac{1}{2n}\sum_{i=1}^{n}\big|gx_i - gu_i\big| \\
&= \frac{1}{2n}\sum_{i=1}^{n} d\big(gx_i, gu_i\big)
\end{aligned}
$$

Hence F *and* g *satisfy the contraction condition (4.1) for* $a_i = \frac{1}{2n}$ *and* $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n}\frac{1}{2n} = \frac{n}{2n} = \frac{1}{2} < 1$*. Also* F *is* g*-monotone mapping. To claim this we have to consider two cases under the condition* $gx \preceq gy$

- *if* $i \in A$*, then* $F(a_1, a_2, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = \left[0, \frac{\sin a_1 - \sin a_2 + \ldots - \sin a_{i-1} + \sin x - \sin a_{i+1} + \ldots + \sin a_n}{4n}\right]$
  $\leq^1 \left[0, \frac{\sin a_1 - \sin a_2 + \ldots - \sin a_{i-1} + \sin y - \sin a_{i+1} + \ldots + \sin a_n}{4n}\right] = F(a_1, a_2, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_n)$,
- *if* $i \in B$*, then* $F(a_1, a_2, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = \left[0, \frac{\sin a_1 - \sin a_2 + \ldots + \sin a_{i-1} - \sin x + \sin a_{i+1} + \ldots + \sin a_n}{4n}\right]$

$$\geq^2 \left[0, \frac{\sin a_1 - \sin a_2 + \ldots + \sin a_{i-1} - \sin y + \sin a_{i+1} + \ldots + \sin a_n}{4n}\right] = F(a_1, a_2, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_n).$$

Thus all conditions of Theorem 2.1 hold and then F and g have one coincidence point $(0, \ldots, 0)$.

3 N-COINCIDENCE POINTS FOR HYBRID PAIR OF MAPPINGS VIA Δ_g-SYMMETRIC PROPERTY

Let (X, d) be a metric space endowed with a partial order $\preceq$. We recall the following definitions.

Definition 3.1. A function $f : X \times X \times \ldots \times X \to \mathbb{R}$ is called lower semi-continuous if and only if for any sequences $\{x^1_m\}_{m\geq 0}$, ..., $\{x^n_m\}_{m\geq 0} \subseteq X$ and $(x^1, \ldots, x^n) \in X^n$, we have

$$\lim_{n\to\infty}(x^1_m, \ldots, x^n_m) = (x^1, \ldots, x^n) \Rightarrow f(x^1, \ldots, x^n) \leq \liminf_{n\to\infty} f(x^1_m, \ldots, x^n_m).$$

Let $T : X \to X$ be a given mapping. We define the set $\Delta_T \subset X^n$ by

$$\Delta_T = \{(x^1, x^2, \ldots, x^n) \in X^n : Tx^i \preceq Tx^j, \texttt{for all } i \in \Lambda_n = \{1, \ldots, n\} \texttt{ and } j \geq i.$$

Definition 3.2. Let $F : X^n \to CL(X)$ be multi-valued with n-variable. We say that F is a Δ_T-symmetric mapping if there exist mappings $\sigma_1, \ldots, \sigma_n : \Lambda_n \to \Lambda_n$ with

$$(x^1, \ldots, x^n) \in \Delta_T \Rightarrow F(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}) \preceq F(x^{\sigma_j(1)}, \ldots, x^{\sigma_j(n)}), \texttt{for all } i \in \Lambda_n \texttt{ and } j \geq i,$$

where, $A \preceq B$ for $A, B \subseteq X$ means that for any element $a \in A$ there exist $b \in B$ such that $a \preceq b$

Example 3.1. *Let $X = \mathbb{R}$, the set of real numbers, with the usual order and metric, i.e., $(X, \leq, d)$, contain an ordered metric space. Define the mappings $F : X^3 \to CL(X)$ and $T : X \to X$ by*

$$\begin{aligned} F(x^1, x^2, x^3) &= \{\frac{x^1}{x^2}, \frac{x^2}{x^3}, \frac{x^1}{x^3}\} \\ T(x) &= \ln x. \end{aligned}$$

Consider $\{\sigma_1, \sigma_2, \sigma_3\}$ be a three permutations from $\Lambda_3 = \{1, 2, 3\}$ into itself in the form:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \ and \ \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Since, $Tx^i \preceq Tx^j$, implies $\frac{x^i}{x^j} \preceq \frac{x^j}{x^i}$ (for, $\ln x^i \preceq \ln x^j \Rightarrow x^i \preceq x^j \Rightarrow (x^i)^2 \preceq (x^j)^2 \Rightarrow 2\ln x^i \preceq 2\ln x^j \Rightarrow \ln x^i - \ln x^j \preceq \ln x^j - \ln x^i$, i.e., $\ln\frac{x^i}{x^j} \preceq \ln\frac{x^j}{x^i}$), then we have $F(x^1, x^2, x^3) = \{\frac{x^1}{x^2}, \frac{x^2}{x^3}, \frac{x^1}{x^3}\} \preceq F(x^2, x^3, x^1) = \{\frac{x^2}{x^3}, \frac{x^3}{x^1}, \frac{x^2}{x^1}\}$ and $F(x^1, x^2, x^3) = \{\frac{x^1}{x^2}, \frac{x^2}{x^3}, \frac{x^1}{x^3}\} \preceq F(x^3, x^1, x^2) = \{\frac{x^3}{x^1}, \frac{x^1}{x^2}, \frac{x^3}{x^2}\}$. Also, the inequalities $\frac{x^2}{x^3} \preceq \frac{x^3}{x^2}$, $\frac{x^3}{x^1} \preceq \frac{x^3}{x^1}$ and $\frac{x^2}{x^1} \preceq \frac{x^3}{x^1}$ implies $F(x^2, x^3, x^1) = \{\frac{x^2}{x^3}, \frac{x^3}{x^1}, \frac{x^2}{x^1}\} \preceq F(x^3, x^1, x^2) = \{\frac{x^3}{x^1}, \frac{x^1}{x^2}, \frac{x^3}{x^2}\}$. Consequently the mapping F is Δ_T- symmetric.

Remark 3.1. Notice that the relation defined on the collection of all closed subsets of X is reflexive and transitive but not anti symmetric, for example consider $X = \mathbb{R}$, $A = \{0, 2\} \subseteq CL(\mathbb{R})$ and $B = \{1, 2\} \subseteq CL(\mathbb{R})$, we have $A \preceq B$ and $B \preceq A$ but $A \neq B$. If we consider that this relation is also anti symmetric, i.e., $A \preceq B$ means that any element $a \in A$ is related with all elements in B , then we can prove the following theorem.

Theorem 3.2. *Let (X,d) be a metric space endowed with a partial order. Suppose that $F : X^n \to CL(X)$ and $g : X \to X$ contain a hybrid pair of single valued mapping with one variable and set valued mapping with n-variables, F has a Δ_g-symmetric property, $g(X)$ is complete, $\Delta_g \neq \emptyset$, $\sigma_1, \dots, \sigma_n$ are any (n) mappings from Λ_n into itself, the function $f : \big[(g(X)\big]^n \to [0,\infty)$ defined for all $x^1, x^2, \dots, x^n \in X$ by*

$$f(gx^1,\dots,gx^n) = \sum_{i=1}^{n} D(gx^i, F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)})) \tag{3.1}$$

is lower semi-continuous and there exists a function $\phi : [0,\infty) \to [a,1), 0 < a < 1$, satisfying

$$\limsup_{r\to t^+} \phi(r) < 1, \quad \texttt{for each } t \in [0,\infty). \tag{3.2}$$

Assume that for any $(x^1,\dots,x^n) \in \Delta_g$, there exist $gu^1,\dots,gu^n \in g(X)$ with $gu^i \in F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)})$, for all $i \in \Lambda_n$ satisfying

$$\sqrt{\phi\big(f(gx^1,\dots,gx^n)\big)}\left[\sum_{i=1}^{n} d(gx^i,gu^i)\right] \le f(gx^1,\dots,gx^n) \tag{3.3}$$

and

$$f(gu^1,\dots,gu^n) \le \phi\big(f(gx^1,\dots,gx^n)\big)\left[\sum_{i=1}^{n} d(gx^i,gu^i)\right]. \tag{3.4}$$

Then F and g have a N-coincidence point, i.e., there exist $(g\xi^1,\dots,g\xi^n) \in \big[g(X)\big]^n$ such that $g\xi^i \in F(\xi^{\sigma_i}(1),\dots,\xi^{\sigma_i}(n))$.

Proof. Since $F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)})$ are well defined and nonempty for all $i \in \Lambda_n$ and for any $(x^1,\dots,x^n) \in X^n$ and $\phi\big(f(gx^1,\dots,gx^n)\big) < 1$, by *the definition of* ϕ, it implies that for $(x^1,\dots,x^n) \in X^n$ we can find $(u^1,\dots,u^n) \in X^n$ with $u^i \in F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)})$ and

$$\sqrt{\phi\big(f(gx^1,\dots,gx^n)\big)}d(gx^i,u^i) \le D\big(gx^i, F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)})\big)$$

for all $i \in \Lambda_n$. By adding the above inequality over all $i \in \Lambda_n$ and using *(3.1)*, one can say that for any $(x^1,\dots,x^n) \in X^n$, there exist $u^i \in F(x^{\sigma_i(1)},\dots,x^{\sigma_i(n)})$ for all i satisfying

$$\sqrt{\phi\big(f(gx^1,\dots,gx^n)\big)}\left[\sum_{i=1}^{n} d(gx^i,u^i)\right] \le f(gx^1,\dots,gx^n).$$

Let $(x_0^1,\dots,x_0^n) \in \Delta_g$ be arbitrary and fixed, this can be done for $\Delta_g \neq \emptyset$. Then there exist $gx_1^i \in F(x_0^{\sigma_i(1)},\dots,x_0^{\sigma_i(n)}),\ \forall\, i$ such that

$$\sqrt{\phi\big(f(gx_0^1,\dots,gx_0^n)\big)}\left[\sum_{i=1}^{n} d(gx_0^i,gx_1^i)\right] \le f(gx_0^1,\dots,gx_0^n) \tag{3.5}$$

and

$$f(gx_1^1,\dots,gx_1^n) \le \phi\big(f(gx_0^1,\dots,gx_0^n)\big)\left[\sum_{i=1}^{n} d(gx_0^i,gx_1^i)\right]. \tag{3.6}$$

From (3.5) and (3.6), we can get

$$\begin{aligned} f(gx_1^1,\dots,gx_1^n) &\le \sqrt{\phi\big(f(gx_0^1,\dots,gx_0^n)\big)}\left(\sqrt{\phi\big(f(gx_0^1,\dots,gx_0^n)\big)}\left[\sum_{i=1}^{n} d(gx_0^i,gx_1^i)\right]\right) \\ &\le \sqrt{\phi\big(f(gx_0^1,\dots,gx_0^n)\big)}f(gx_0^1,\dots,gx_0^n). \end{aligned}$$

Since F *is* Δ_g*-symmetric mapping* and $(x_0^1,\dots,x_0^n)\in\Delta_g$ then we have

$$F(x_0^{\sigma_i(1)},\dots,x_0^{\sigma_i(n)}) \preceq F(x_0^{\sigma_j(1)},\dots,x_0^{\sigma_j(n)}),\texttt{for all } i\in\Lambda_n \texttt{ and } j\geq i.$$

Thus

$$gx_1^i \preceq gx_1^j,\texttt{for all } i\in\Lambda_n \texttt{ and } j\geq i\Rightarrow(x_1^1,\dots,x_1^n)\in\Delta_g.$$

Again by (3.3)and (3.4), we can find $(x_2^1,\dots,x_2^n)\in X^n$, $gx_2^i\in F(x_1^{\sigma_i^1},\dots,x_1^{\sigma_i^n})$ for all i such that

$$\sqrt{\phi\big(f(gx_1^1,\dots,gx_1^n)\big)}\left[\sum_{i=1}^n d(gx_1^i,gx_2^i)\right]\leq f(gx_1^1,\dots,gx_1^n)$$

and

$$f(gx_2^1,\dots,gx_2^n)\leq\phi\big(f(gx_1^1,\dots,gx_1^n)\big)\left[\sum_{i=1}^n d(gx_1^i,gx_2^i)\right].$$

Hence, we get

$$f(gx_2^1,\dots,gx_2^n)\leq\sqrt{\phi\big(f(gx_1^1,\dots,gx_1^n)\big)}f(gx_1^1,\dots,gx_1^n),\ \texttt{with } (x_2^1,\dots,x_2^n)\in\Delta_g.$$

Continuing this process we can construct sequences $\{x_m^1\}_{m\geq0},\dots,\{x_m^n\}_{m\geq0}\in X$, such that $(x_m^1,\dots,x_m^n)\in\Delta_g$, $gx_{m+1}^i\in F(x_m^{\sigma_i^1},\dots,x_m^{\sigma_i^n})\ \forall\ i,m,$

$$\sqrt{\phi\big(f(gx_m^1,\dots,gx_m^n)\big)}\left[\sum_{i=1}^n d(gx_m^i,gx_{m+1}^i)\right]\leq f(gx_m^1,\dots,gx_m^n) \tag{3.7}$$

and

$$f(gx_{m+1}^1,\dots,gx_{m+1}^n)\leq\sqrt{\phi\big(f(gx_m^1,\dots,gx_m^n)\big)}f(gx_m^1,\dots,gx_m^n),\ \texttt{with } (x_{m+1}^1,\dots,x_{m+1}^n)\in\Delta_g. \tag{3.8}$$

Now, we shall show that $f(gx_m^1,\dots,gx_m^n)\to0,\ \ as\ \ m\to\infty$. If $f(gx_m^1,\dots,gx_m^n)=0$ for some m then we have $\sum_{i=1}^n D(gx_m^i,F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(n)}))=0\Rightarrow D(gx_m^i,F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(n)}))=0$ for all i, which implies that $gx_m^i\in F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(n)})$ or $gx_m^i\in\overline{F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(n)})}=F(x_m^{\sigma_i(1)},\dots,x_m^{\sigma_i(n)})$ for all i, i.e., $(gx_m^1,\dots,gx_m^n)$ is a N-coincidence point of F and g and the proof is completed. So, we shall assume that $f(gx_m^1,\dots,gx_m^n)>0$ for all $m\geq0$.

By (3.8) and $\phi(t)<1$, we conclude that $\{f(gx_m^1,\dots,gx_m^n)\}$ is strictly decreasing sequence of positive real numbers then we can find $\delta\geq0$ such that

$$\lim_{m\to\infty}f(gx_m^1,\dots,gx_m^n)=\delta.$$

Now, we shall prove that $\delta=0$. On contrary, assume that $\delta>0$. Letting $m\to\infty$ in (3.8) and using (3.2) yield

$$\delta\leq\limsup_{f(gx_m^1,\dots,gx_m^n)\to\delta^+}\sqrt{\phi\big(f(gx_m^1,\dots,gx_m^n)\big)}\delta<\delta.$$

But this is a contradiction. Hence $\delta=0$ and $\lim_{m\to\infty}f(gx_m^1,\dots,gx_m^n)=0$.
Then, we will prove that $\{gx_m^i\}_{m\geq0}$ are cauchy sequences for all $i\in\Lambda_n$. Suppose that

$$\alpha=\limsup_{m\to\infty}\sqrt{\phi\big(f(gx_m^1,\dots,gx_m^n)\big)}<1.$$

Let, k be such that $\alpha<k<1$, then there exist $m_0\in\mathbb{N}$ such that

$$\sqrt{\phi\big(f(gx_m^1,\dots,gx_m^n)\big)}<k,\ \texttt{for all } m\geq m_0.$$

Therefore

$$\begin{aligned} f(gx^1_{m+1},\dots,gx^n_{m+1}) &\le \sqrt{\phi\big(f(gx^1_m,\dots,gx^n_m)\big)}f(gx^1_m,\dots,gx^n_m)\\ &\le kf(gx^1_m,\dots,gx^n_m)\\ &\le k^2f(gx^1_{m-1},\dots,gx^n_{m-1}) \qquad (3.9)\\ &\vdots\\ &\le k^{m+1-m_0}f(gx^1_{m_0},\dots,gx^n_{m_0}). \end{aligned}$$

Since $\phi(t)\ge a>0$ for all $t\ge 0$, from (3.7) and (3.9), we obtain

$$\left[\sum_{i=1}^n d(gx^i_m,gx^i_{m+1})\right] < \frac{1}{\sqrt{a}}k^{m-m_0}f(gx^1_{m_0},\dots,gx^n_{m_0}),\forall\, m>m_0.$$

Now, let use consider the following for fixed i

$$\begin{aligned} \sum_{i=1}^n d(gx^i_m,gx^i_{m+p}) &\le \sum_{i=1}^n\left[d(gx^i_m,gx^i_{m+1})+d(gx^i_{m+1},gx^i_{m+2})+\ldots+d(gx^i_{m+p-1},gx^i_{m+p})\right]\\ &\le \frac{1}{\sqrt{a}}k^{m-m_0}f(gx^1_{m_0},\dots,gx^n_{m_0})\\ &+\frac{1}{\sqrt{a}}k^{m+1-m_0}f(gx^1_{m_0},\dots,gx^n_{m_0})\\ &\vdots\\ &+\frac{1}{\sqrt{a}}k^{m+p-1-m_0}f(gx^1_{m_0},\dots,gx^n_{m_0})\\ &\le \frac{1}{\sqrt{a}}[k^{m-m_0}+k^{m+1-m_0}+\ldots+k^{m+p-1-m_0}]f(gx^1_{m_0},\dots,gx^n_{m_0})\\ &\le \frac{1}{\sqrt{a}}\frac{k^{m-m_0}(1-k^p)}{1-k}f(gx^1_{m_0},\dots,gx^n_{m_0})\\ &\le \frac{1}{\sqrt{a}}\frac{k^{m-m_0}}{1-k}f(gx^1_{m_0},\dots,gx^n_{m_0})\to 0 \ \texttt{as}\ \ m\to\infty, \end{aligned}$$

which yields that $\{\{gx^i_m\},1\le i\le n\}$ are Cauchy sequences in $g(X)$, which is complete then there exist $(u^1,\dots,u^n)\in g(X)$ $gx^i_m\to u^i=g\xi^i$ for all $i\in\Lambda_n$.

Finally, we show that $(g\xi^1,\dots,g\xi^n)$ is a N-coincidence point for F and g. Using the lower semi-continuity of f we get

$$\begin{aligned} 0\le f(g\xi^1,\dots,g\xi^n) &= \sum_{i=1}^n D(g\xi^i,F(\xi^{\sigma_i(1)},\dots,\sigma_i(n)))\\ &\le \liminf_{m\to\infty} f(gx^1_m,\dots,gx^n_m)=0. \end{aligned}$$

Hence, $D(g\xi^i,F(\xi^{\sigma_i(1)},\dots,\sigma_i(n)))=0$ and $g\xi^i\in F(\xi^{\sigma_i(1)},\dots,\sigma_i(n))$ for all i, i.e., $(\xi^i)^n_{i=1}$ is N-coincidence point for F and g. □

Theorem 3.3. *Let (X,d) be a metric space endowed with a partial order. Suppose that $F:X^n\to CL(X)$ and $g:X\to X$ contain a hybrid pair of single valued mapping with one variable and set valued mapping with n-variables, F has a Δ_g-symmetric property, $g(X)$ is complete, $\Delta_g\neq\emptyset$,*

$\sigma_1, \ldots, \sigma_n$ are any (n) mappings from Λ_n into itself, the function $f\big[(g(X)\big]^n \to [0,\infty)$ defined for all $x^1, x^2, \ldots, x^n \in X$ by

$$f(gx^1, \ldots, gx^n) = \sum_{i=1}^{n} D(gx^i, F(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)})) \tag{3.10}$$

is lower semi-continuous and there exists a function $\phi : [0,\infty) \to [a,1), 0 < a < 1,$ satisfying

$$\limsup_{r \to t^+} \phi(r) < 1, \quad \textit{for each } t \in [0,\infty). \tag{3.11}$$

Assume that for any $(x^1, \ldots, x^n) \in \Delta_g$ there exist $(gu^1, \ldots, gu^n) \in \big[g(X)\big]^n$ with $gu^i \in F(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)})$, for all $i \in \Lambda_n$ satisfying

$$\sqrt{\phi\Big(\sum_{i=1}^{n} d(gx^i, gu^i)\Big)}\Big[\sum_{i=1}^{n} d(gx^i, gu^i)\Big] \leq f(gx^1, \ldots, gx^n) \tag{3.12}$$

and

$$f(gu^1, \ldots, gu^n) \leq \phi\Big(\sum_{i=1}^{n} d(gx^i, gu^i)\Big)\Big[\sum_{i=1}^{n} d(gx^i, gu^i)\Big]. \tag{3.13}$$

Then F and g have a N-coincidence point, i.e., there exist $(g\xi^1, \ldots, g\xi^n) \in \big[g(X)\big]^n$ such that $g\xi^i \in F(\xi^{\sigma_i}(1), \ldots, \xi^{\sigma_i}(n))$.

Proof. Let $(x_0^1, \ldots, x_0^n) \in \Delta_g$ as in Theorem 3.2, we can find $gx_1^i \in F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)}),\ \forall\ i$ such that

$$\sqrt{\phi(\triangle_0)}\triangle_0 \leq f(gx_0^1, \ldots, gx_0^n) \tag{3.14}$$

and

$$f(gx_1^1, \ldots, gx_1^n) \leq \phi(\triangle_0)\triangle_0, \tag{3.15}$$

where, $\triangle_0 = \sum_{i=1}^{n} d(gx_0^i, gx_1^i)$. From (3.14) and (3.15), we can get

$$\begin{aligned} f(gx_1^1, \ldots, gx_1^n) &\leq \sqrt{\phi(\triangle_0)}\Big(\sqrt{\phi(\triangle_0)}\triangle_0\Big) \\ &\leq \sqrt{\phi(\triangle_0)} f(gx_0^1, \ldots, gx_0^n) \end{aligned}$$

and using the Δ_g-symmetric property of F implies $(x_1^1, \ldots, x_1^n) \in \Delta_g$.

Consequently, we can construct sequences $\{x_m^1\}_{m\geq 0}, \ldots, \{x_m^n\}_{m \geq 0} \in X$ such that $(x_m^1, \ldots, x_m^n) \in \Delta_g$, $gx_{m+1}^i \in F(x_m^{\sigma_i^1}, \ldots, x_m^{\sigma_i^n})\ \ \forall\ \ i, m,$

$$\sqrt{\phi(\triangle_m)}\triangle_m \leq f(gx_m^1, \ldots, gx_m^n) \tag{3.16}$$

and

$$f(gx_{m+1}^1, \ldots, gx_{m+1}^n) \leq \sqrt{\phi(\triangle_m)} f(gx_m^1, \ldots, gx_m^n), \ \ \texttt{with}\ \ (x_{m+1}^1, \ldots, x_{m+1}^n) \in \Delta_g, \tag{3.17}$$

where, $\triangle_m = \sum_{i=1}^{n} d(gx_m^i, gx_{m+1}^i)$. By (3.17) and $\phi(t) < 1$, we conclude that $\big\{f(gx_m^1, \ldots, gx_m^n)\big\}$ is strictly decreasing sequence of positive real numbers then we can find $\delta \geq 0$ such that

$$\lim_{m \to \infty} f(gx_m^1, \ldots, gx_m^n) = \delta. \tag{3.18}$$

To prove that $\delta = 0$ we have to show that $\{\triangle_m\}_{m\geq 0}$ admits a subsequence converging to θ^+ for some $\theta \geq 0$. From $\phi(t) \geq a > 0$ for all $t \in [0, \infty)$ and (3.16), we obtain

$$0 \leq \triangle_m \leq \frac{1}{\sqrt{a}} f(gx_m^1, \dots, gx_m^n) \leq \frac{1}{\sqrt{a}} f(gx_0^1, \dots, gx_0^n). \tag{3.19}$$

It is obvious that $\triangle_m$ is bounded sequence of non-negative real numbers. Since each bounded sequence $\triangle_m$ of real numbers has a monotone subsequence which is also bounded and every bounded sequence is convergent, then this is guarantee the existence of subsequence that converge. Therefore, we know the collection E of all subsequential limits (limits of all convergent subsequences) is non-empty, i.e., the lower limit $\liminf_{m\to\infty} \triangle_m = \inf E$ and the upper limit $\limsup_{m\to\infty} \triangle_m = \sup E$ exist. Thus, there exist $\theta \geq 0$ with

$$\liminf_{m\to\infty} \triangle_m = \theta. \tag{3.20}$$

Since $gx_{m+1}^i \in F(x_m^{\sigma_i(1)}, \dots, x_m^{\sigma_i(n)})$, we have

$$\triangle_m = \sum_{i=1}^{n} d(gx_m^i, gx_{m+1}^i) \geq \sum_{i=1}^{n} D(gx_m^i, F(x_m^{\sigma_i(1)}, \dots, x_m^{\sigma_i(n)})) = f(gx_m^1, \dots, gx_m^n). \tag{3.21}$$

Taking the upper limit as m tends to infinity in Equation (3.21) and using (3.18) and (3.20) yields

$$\liminf_{m\to\infty} \triangle_m \geq \liminf_{m\to\infty} f(gx_m^1, \dots, gx_m^n)$$
$$\theta \geq \delta.$$

Now, we shall show that $\theta = \delta$. If $\delta = 0$, by (3.18), (3.19) and (3.20), we obtain

$$\begin{aligned} 0 &\leq \liminf_{m\to\infty} \triangle_m \leq \frac{1}{\sqrt{a}} \lim_{m\to\infty} f(gx_m^1, \dots, gx_m^n) \\ 0 &\leq \theta \leq \frac{1}{\sqrt{a}} \delta = 0 \\ &\Rightarrow \theta = 0. \end{aligned}$$

Consider $\delta > 0$ and we want to claim that $\theta = \delta$. On contrary assume that $\theta > \delta$ and $\theta - \delta > 0$. From (3.18), for any $\epsilon = \frac{\theta-\delta}{4}$ there exist a positive integer m_1, such that

$$|f(gx_m^1, \dots, gx_m^n) - \delta| < \epsilon \texttt{ for all } m > m_1$$

Also from (3.20), for the same $\epsilon > 0$, we can find $m_2 \in \mathbb{N}$ such that

$$\theta - \epsilon < \triangle_m \texttt{ for all } m > m_2.$$

Hence we have

$$f(gx_m^1, \dots, gx_m^n) < \delta + \frac{\theta - \delta}{4}$$

and

$$\theta - \frac{\theta - \delta}{4} < \triangle_m \texttt{ for all } m > m_0 = \max\{m_1, m_2\}.$$

Using (3.16)

$$\sqrt{\phi(\triangle_m)}(\theta - \frac{\theta - \delta}{4}) < \sqrt{\phi(\triangle_m)}\triangle_m \leq f(gx_m^1, \dots, gx_m^n) < \delta + \frac{\theta - \delta}{4}.$$

Therefore

$$\sqrt{\phi(\triangle_m)} < \frac{\theta+3\delta}{\delta+3\theta}, \quad \texttt{for all} \ \ m > m_0. \tag{3.22}$$

Putting $h = \frac{\theta+3\delta}{\delta+3\theta} < 1$ in (3.22) and using (3.17) imply

$$\begin{aligned} f(gx^1_{m+1}, \ldots, gx^n_{m+1}) &\leq \sqrt{\phi(\triangle_m)} f(gx^1_m, \ldots, gx^n_m) \\ &< h f(gx^1_m, \ldots, gx^n_m) \\ &< h^2 f(gx^1_{m-1}, \ldots, gx^n_{m-1}) \\ &\vdots \\ &< h^{m+1-m_0} f(gx^1_{m_0}, \ldots, gx^n_{m_0}) \ \ \texttt{for all} \ \ m > m_0. \end{aligned}$$

Therefore for $m = m_0 + k_0$, we have

$$\begin{aligned} \delta \leq f(gx^1_{m_0+k_0}, \ldots, gx^n_{m_0+k_0}) &< h^{k_0} f(gx^1_{m_0}, \ldots, gx^n_{m_0}) \\ &< \delta. \end{aligned}$$

That is a contradiction, then $\theta = \delta$.

$$\begin{aligned} \theta = \delta \leq f(gx^1_m, \ldots, gx^n_m) &\leq \triangle_m \\ \Rightarrow \theta \leq \triangle_m \ \ &\texttt{for all} \ \ m \geq 0. \end{aligned}$$

Then, we rewrite

$$\liminf_{m\to\infty} \triangle_m = \theta^+.$$

That means, the sequence $\{\triangle_m\}_{m\geq 0}$ contains a subsequence, say $\{\triangle_{m_k}\}_{k\geq 0}$, that converge to θ^+. By (3.11), we have

$$\limsup_{\triangle_{m_k}\to\theta^+} \sqrt{\phi(\triangle_{m_k})} < 1. \tag{3.23}$$

and from (3.17)

$$f(gx^1_{m_k+1}, \ldots, gx^n_{m_k+1}) \leq \sqrt{\phi(\triangle_{m_k})} f(gx^1_{m_k}, \ldots, gx^n_{m_k}).$$

Now, we want to show that $\delta = 0$. Assume on contrary that $\delta > 0$. By taking the upper limit above as $k \to \infty$, we can obtain a contradiction and then we conclude that $\delta = 0$ as follows

$$\delta = \limsup_{k\to\infty} f(gx^1_{m_k+1}, \ldots, gx^n_{m_k+1}) \leq \limsup_{\triangle_{m_k}\to\theta^+} \sqrt{\phi(\triangle_{m_k})} \limsup_{k\to\infty} f(gx^1_{m_k}, \ldots, gx^n_{m_k}) < 1.\delta.$$

Hence $\delta = 0$ and $\lim_{m\to\infty} f(gx^1_m, \ldots, gx^n_m) = 0$.
Then, we will prove that $\{gx^i_m\}_{m\geq 0}$ are cauchy sequences for all $i \in \Lambda_n$. Suppose that

$$\alpha = \limsup_{\triangle_{m_k}\to 0^+} \sqrt{\phi(\triangle_{m_k})} < 1.$$

Consider k be such that $\alpha < k < 1$, then there exist finitely many $\sqrt{\phi(\triangle_{m_k})}$, say $\{\sqrt{\phi(\triangle_{m_{k_1}})}, \ldots, \sqrt{\phi(\triangle_{m_{k_n}})}\}$, such that $\sqrt{\phi(\triangle_{m_k})} \geq k$. That means, we can find $m_0 = m_{k_{n+1}} \in \mathbb{N}$ with

$$\sqrt{\phi(\triangle_{m_k})} < k, \quad \texttt{for all} \ \ m_k \geq m_0.$$

For simplify, we set $m_k = p$.

$$\begin{aligned} f(gx^1_{p+1}, \ldots, gx^n_{p+1}) &\leq \sqrt{\phi(\triangle_p)} f(gx^1_p, \ldots, gx^n_p) \\ &\leq k f(gx^1_p, \ldots, gx^n_p) \\ &\leq k^2 f(gx^1_{p-1}, \ldots, gx^n_{p-1}) \\ &\vdots \\ &\leq k^{p+1-m_0} f(gx^1_{m_0}, \ldots, gx^n_{m_0}). \end{aligned} \tag{3.24}$$

Since $\phi(t) \geq a > 0$ for all $t \geq 0$, from (3.16) and (3.24), we obtain

$$\left[\sum_{i=1}^{n} d(gx^i_p, gx^i_{p+1})\right] < \frac{1}{\sqrt{a}} k^{p-m_0} f(gx^1_{m_0}, \ldots, gx^n_{m_0}), \forall\ m > m_0.$$

Now, let use consider the following for fixed i

$$\begin{aligned} \sum_{i=1}^{n} d(gx^i_p, gx^i_{p+q}) &\leq \sum_{i=1}^{n} \left[d(gx^i_p, gx^i_{p+1}) + d(gx^i_{p+1}, gx^i_{p+2}) + \ldots + d(gx^i_{p+q-1}, gx^i_{p+q})\right] \\ &\leq \frac{1}{\sqrt{a}} k^{p-m_0} f(gx^1_{m_0}, \ldots, gx^n_{m_0}) \\ &+ \frac{1}{\sqrt{a}} k^{p+1-m_0} f(gx^1_{m_0}, \ldots, gx^n_{m_0}) \\ &\vdots \\ &+ \frac{1}{\sqrt{a}} k^{p+q-1-m_0} f(gx^1_{m_0}, \ldots, gx^n_{m_0}) \\ &\leq \frac{1}{\sqrt{a}} [k^{p-m_0} + k^{p+1-m_0} + \ldots + k^{p+q-1-m_0}] f(gx^1_{m_0}, \ldots, gx^n_{m_0}) \\ &\leq \frac{1}{\sqrt{a}} \frac{k^{p-m_0}(1-k^q)}{1-k} f(gx^1_{m_0}, \ldots, gx^n_{m_0}) \\ &\leq \frac{1}{\sqrt{a}} \frac{k^{p-m_0}}{1-k} f(gx^1_{m_0}, \ldots, gx^n_{m_0}) \to 0 \text{ as } p \to \infty. \end{aligned}$$

Which yields that $\{\{gx^i_m\}, 1 \leq i \leq n\}$ are Cauchy sequences in $g(X)$, which is complete then there exist $(u^1, \ldots, u^n) \in g(X)$ $gx^i_m \to u^i = g\xi^i$ for all $i \in \Lambda_n$.

Finally, we show that $(g\xi^1, \ldots, g\xi^n)$ is a n-coincidence point for F and g. Using the lower semi-continuity of f, we get

$$\begin{aligned} 0 \leq f(g\xi^1, \ldots, g\xi^n) &= \sum_{i=1}^{n} D(g\xi^i, F(\xi^{\sigma_i(1)}, \ldots, \sigma_i(n))) \\ &\leq \liminf_{m\to\infty} f(gx^1_m, \ldots, gx^n_m) = 0. \end{aligned}$$

Hence, $D(g\xi^i, F(\xi^{\sigma_i(1)}, \ldots, \sigma_i(n))) = 0$ and $g\xi^i \in F(\xi^{\sigma_i(1)}, \ldots, \sigma_i(n))$ for all i, i.e., $(\xi^i)^n_{i=1}$ is N-coincidence point for F and g. $\square$

4 CONCLUSION

This Paper aimed to study three coincidence point theorems for hybrid pair of mappings one of them single-valued mapping $g : X \to X$ and another one is multi-valued mapping with n-variable $F : X^n \to CL(X)$ in partially ordered metric spaces $(X, d, \preceq)$, not necessarily complete. As special case of Theorem 2.1 , if we consider that the mapping F is single valued instead of multi-valued, we will obtain the following corollary.

Corollary 4.1. *Let $(X, \preceq, d)$ be an ordered metric space and $\phi = (\sigma_1, \ldots, \sigma_n)$ be a (n)-tuple of mappings from $\Lambda_n = \{1, 2, \ldots, n\}$ into itself such that, $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega_{\acute{A},B}$ if $i \in B$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property, $F(X^n) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Assume that there exist $a_i \in R, i \in \Lambda_n$ verifying $\sum_{i=1}^{n} a_i < 1$ and*

$$d\big(F(x^1, \ldots, x^n), F(y^1, \ldots, y^n)\big) \leq \sum_{j=1}^{n} a_j d(gx^j, gy^j) \tag{4.1}$$

for which $gx^i \preceq_i gy^i$. If there exist $x_0^1, \ldots x_0^n \in X$ such that

$$g(x_0^i) \preceq_i F(x_0^{\sigma_i(1)}, \ldots, x_0^{\sigma_i(n)}), \text{ for all } i \in \Lambda_n \tag{4.2}$$

and X has the sequential g-monotone property. Then F and g have, at least, one N-coincidence point.

Also in section three we used Δ_g- symmetric property to extend the main result of Samet and vetro to any number of variables and obtain the corresponding N- coincidence point theorem.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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