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Maximum Product of Spacing Parameter Estimation of Gompertz Rayleigh Distribution and Application to Rainfall Datasets

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Authors' contributions

This work was carried out in collaboration among all authors. Author HAA designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors SOB and AMI managed the analyses of the study. Author YAO managed the literature searches. All authors read and approved the final manuscript.

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Abstract

Gompertz Rayleigh (GomR) distribution was introduced in an earlier study with few statistical properties derived and parameters estimated using only the most common traditional method, Maximum Likelihood Estimation (MLE). This paper aimed at deriving more statistical properties of the GomR distribution, estimating the three unknown parameters via a competitive method, Maximum Product of Spacing (MPS) and evaluating goodness of fit using rainfall data sets from Nigeria, Malaysia and Argentina.

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Properties of statistical distributions including distribution of smallest and largest order statistics, cumulative or integrated hazard function, odds function, r^{th} non-central moments, moment generating function, mean, variance and entropy measures for GomR distribution were explicitly derived. The fitted data sets reveal the flexibility of GomR distribution over other distributions been compared with. Simulation study was used to evaluate the consistency, accuracy and unbiasedness of the GomR distribution parameter estimates obtained from the method of MPS. The study found that GomR distribution could not provide a better fit for Argentine rainfall data but it was the best distribution for the rainfall data sets from Nigeria and Malaysia in comparison with the distributions; Generalized Weibull Rayleigh (GWR), Exponentiated Weibull Rayleigh (EWR), Type (II) Topp Leone Generalized Inverse Rayleigh (TIITLGIR), Kumarawamy Exponential Inverse Raylrigh (KEIR), Negative Binomial Marshall-Olkin Rayleigh (NBMOR) and Exponentiated Weibull (EW). Furthermore, the estimates from MPSE were consistent as the sample size increases but not as efficient as those from MLE.

Keywords: Gompertz-Rayleigh; probability distribution; smallest and largest order statistics; entropy measures; maximum product of spacing.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16.

1 Introduction

Rayleigh(R) is a continuous lifetime distribution introduced by [1], it has few properties of the Weibull (W) distribution. Areas including communication theory, medical imaging science, engineering among others benefits from this distribution. Compared to other widely used classical distributions, fewer source of materials were found on R distribution.

Several modification of R exist in literature. The inverted form, Inverse Rayleigh (IR) was introduced by [2] which have some properties of the Inverse Weibull (IW) distribution. [3] and [4] respectively studied maximum likelihood estimator, percentile estimator and different estimation methods of the parameter of IR. Ali et al., 2015 studied and found that "higher order statistics as well as the variance of the IR do not exist. [5] introduced the generalized Rayleigh also known as exponentiated rayleigh (ER) having some properties of log-normal distribution and is a special case of exponentiated weibull by [6]. Other modifications include transmuted Weibull Rayleigh by [7], Gompertz Rayleigh by [8], Power Rayleigh by [9] and recent addition in literature are Inverse Weibull Rayleigh by [10] and extended odd Weibull Rayleigh by [11].

The Maximum Likelihood estimation (MLE) is the most frequently used method of estimation not just in statistical distributions but in statistics as a whole because of its desirable properties. The estimates are obtained by maximizing the likelihood function of the PDF. This method however have it setbacks making the estimators fail sometimes.

Maximum Product of Spacing estimation (MPS), introduced by [12] is likely to serve as a competitor of MLE in cases where the estimates from MLE breaks down. The estimators obtained by maximizing the geometric mean of spacings between cumulative distribution function in close observations are consistent and as efficient as MLE . [13] noted that " The MLE perfectly estimates the parameters of discrete distributions if the contribution to the likelihood function is bounded from above but not for compound continuous distributions". The consistency of MPS was studied and shown that it works in place of MLE.

[14] introduced Gompertz Exponentiated Rayleigh distribution and adopted MPS method. The study found estimates of most parameters from MPS more efficient than those from ML especially at larger sample sizes. More-so, [15] estimates the parameters of newly introduced Marshall-Olkin Alpha Power Lomax distribution using MPS along with Least Squares Estimation (LSE) and MLE. The study reveal that all methods were consistent and efficient, but those from LSE have more relative efficiency for most parameters.

There exist plethora of extended forms of R distributions, however, the traditional method, Maximum likelihood estimation is mostly used to estimate their parameters. The parameters of extended R distributions namely Gompertz Rayleigh (GomR) by [8] were estimated using only MLE. Estimation these parameters using addition method would be very important to reliability and applied statisticians.

Moreso, no extended distribution in literature thus far have utilize the rainfall data sets from these regions to access their goodness of fit.

2 Gompertz-Rayleigh (GomR) Distribution

Using a family of distribution Gompertz-G by [16], [8] defined the CDF and PDF of GomR distribution as follows

$$\begin{aligned} F(x;\alpha,\beta,\gamma) &= 1-e^{\frac{\alpha}{\beta}(1-[1-(1-e^{\frac{-x^2}{2\gamma^2}})]^{-\beta})} \\ &= 1-e^{\frac{\alpha}{\beta}(1-e^{\frac{x^2\beta}{2\gamma^2}})} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} f(x;\alpha,\beta,\gamma) &= \frac{\alpha x}{\gamma^2}e^{\frac{-x^2}{2\gamma^2}}[1-(1-e^{\frac{-x^2}{2\gamma^2}})]^{-\beta-1}e^{\frac{\alpha}{\beta}\{1-[1-(1-e^{\frac{-x^2}{2\gamma^2}})]^{-\beta}\}} \\ &= \frac{\alpha x e^{\frac{-x^2}{2\gamma^2}}}{\gamma^2}[e^{\frac{-x^2}{2\gamma^2}}]^{-\beta-1}e^{\frac{\alpha}{\beta}[1-e^{\frac{x^2\beta}{2\gamma^2}}]} \\ &= \frac{\alpha x}{\gamma^2}\, e^{\frac{\beta x^2}{2\gamma^2}}\, e^{\frac{\alpha}{\beta}[1-e^{\frac{x^2\beta}{2\gamma^2}}]} \end{aligned} \tag{2.2}$$

where $\alpha, \beta > 0$ are shape parameters while $\gamma > 0$ is a scale parameter and $x > 0$.

2.1 Useful representation

The CDF and PDF of GomR distribution can be represented in simpler as follows.
Using

$$e^x = \sum_{b_1=0}^{\infty} \frac{x^i}{b_1!}$$

and

$$(1-m)^b = \sum_{b_2=1}^{\infty} \binom{b}{b_2} (-1)^{b_2} x^{b_2}$$

the simplified CDF is

$$\begin{aligned} F(x) &= 1-\sum_{b_1=0}^{\infty}\frac{\left(\frac{\alpha}{\beta}\right)^{b_1}\left[1-e^{\frac{x^2\beta}{2\gamma^2}}\right]^{b_1}}{b_1!} \\ &= 1-\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\frac{1}{a_1!}(-1)^{b_2}\left(\frac{\alpha}{\beta}\right)^{a_1}\binom{b_1}{b_2}e^{\frac{x^2\beta b_2}{2\gamma^2}} \end{aligned} \quad (2.3)$$

the simplified PDF is

$$\begin{aligned} f(x) &= \frac{\alpha x}{\gamma^2}e^{\frac{x^2\beta}{2\gamma^2}}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\frac{1}{a_1!}(-1)^{b_2}\left(\frac{\alpha}{\beta}\right)^{a_1}\binom{b_1}{b_2}e^{\frac{x^2\beta b_2}{2\gamma^2}} \\ &= \frac{\alpha x}{\gamma^2}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\frac{1}{b_1!}(-1)^{b_2}\left(\frac{\alpha}{\beta}\right)^{b_1}\binom{b_1}{b_2}e^{\frac{x^2\beta(1+b_2)}{2\gamma^2}} \end{aligned} \quad (2.4)$$

3 Properties of GomR Distribution

[8] presented the shapes of the densities, hazard and survival function of the GomR distribution for several parameter values. GomR distribution have positively skewed density function, an increasing hazard function and a decreasing survival function. The study further derived the reliability and quantile function.

4 Additional Statistical Properties of GomR Distribution

4.1 Order statistic of the GomR distribution

Given a distribution with sample of independent characteristics $X_1, X_2, \ldots, X_n$ .This can be represented in a notation $X_{(1,n)} \le X_{(2,n)} \le \cdots \le X_{(n,n)}$ or ordered as $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$. $X_{(1)}$ representing the 1^{st} order is considered the minimum , $X_{(2)}$ representing the 2^{nd} order, the second minimum while the n^{th} order statistics, $X_{(n)}$, is the maximum.

4.1.1 PDF of the k^{th} order statistics of GomR distribution

Suppose a random sample $X_1, X_2, \ldots, X_n$ is obtained from the densities of the GomR distribution and represents $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ as the order statistic, then the PDF, $f_{(k,n)}(x)$, the k^{th} order statistics is expressed as

$$f_{(k,n)}(x) = \frac{n!}{(k-1)!(n-k)!}f(x)\times F(x)^{k-1}\times[1-F(x)]^{n-k} \quad (4.1)$$

where F(x) and f(x) are the CDF and PDF of the GomR distribution.

For easier simplification, the binomial expansion on $[1-F(x)]^{n-k}$ was used

$$[1-F(x)]^{n-k} = \sum_{b_3=0}^{\infty}\binom{n-k}{b_3}(-1)^{b_3}(F(x))^{b_3} \quad (4.2)$$

Substituting (4.2) in (4.1) results to

$$f_{(k,n)}(x) = \sum_{b_3=0}^{\infty}\frac{n!}{(k-1)!\,(n-k)!}\,f(x)\,.\binom{n-k}{b_3}(-1)^{b_3}(F(x))^{b_3+k-1} \quad (4.3)$$

Again, substituting (2.1) and (2.2) in (4.3), gives the k^{th} order statistic of the GomR distribution as

$$f_{(k,n)}(x) = \sum_{b_3=0}^{\infty} \frac{n!\,(-1)^{b_3}}{(k-1)!\,(n-k-b_3)!\;b_3!}\;\frac{\alpha x}{\gamma^2}\,e^{\frac{\beta x^2}{2\gamma^2}}\;e^{\frac{\alpha}{\beta}[1-e^{\frac{x^2\beta}{2\gamma^2}}]} \times \left[1-e^{\frac{\alpha}{\beta}(1-e^{\frac{x^2\beta}{2\gamma^2}})}\right]^{b_3+k-1} \tag{4.4}$$

4.1.2 PDF of the smallest and largest order statistics

Substituting k=1 into 4.4 gives the PDF of minimum or 1st order statistics as

$$f_{(1,n)}(x) = \sum_{b_3=0}^{\infty} \frac{n!\,(-1)^{b_3}}{(n-1-b_3)!\;b_3!}\;\frac{\alpha x}{\gamma^2}\,e^{\frac{\beta x^2}{2\gamma^2}}\;e^{\frac{\alpha}{\beta}[1-e^{\frac{x^2\beta}{2\gamma^2}}]} \times \left[1-e^{\frac{\alpha}{\beta}(1-e^{\frac{x^2\beta}{2\gamma^2}})}\right]^{b_3} \tag{4.5}$$

Similarly, the n^{th} order or the maximum order statistics was obtained by substituting k=n as

$$f_{(n,n)}(x) = \sum_{b_3=0}^{\infty} \frac{n!\,(-1)^{b_3}}{(n-k)!\,(-b_3)!\;b_3!}\;\frac{\alpha x}{\gamma^2}\,e^{\frac{\beta x^2}{2\gamma^2}}\;e^{\frac{\alpha}{\beta}[1-e^{\frac{x^2\beta}{2\gamma^2}}]} \times \left[1-e^{\frac{\alpha}{\beta}(1-e^{\frac{x^2\beta}{2\gamma^2}})}\right]^{b_3+n-1} \tag{4.6}$$

4.2 Reliability analysis of GO-R distribution

Suppose a random variable X comes from the GomR distribution with PDF, f(x) and CDF, F(x); the following properties were obtained.

4.2.1 Cumulative or integrated hazard function

This is a risk function and not a probability. The cumulative hazard function of GomR is derived as follows

$$\begin{aligned} H(x) &= \int_0^t h(x)dx \\ &= \frac{\alpha}{\gamma^2}\int_0^t x\,e^{\frac{\beta\,x^2}{2\,\gamma^2}}\,dx \end{aligned} \tag{4.7}$$

$$let \quad \frac{x^2\,\beta}{2\,\gamma^2}, \quad then \quad \frac{du}{dx} = \frac{x\,\beta}{\gamma^2} \quad and \quad dx = \frac{\gamma^2\,du}{x\,\beta}$$

$$Now, \qquad x \to 0, \quad u \to 0 \qquad and \qquad x \to t, \qquad u \to \frac{t^2\,\beta}{2\,\gamma^2}$$

$$\begin{aligned} H(x) &= \frac{\alpha}{\beta}\int_0^{\frac{t^2\,\beta}{2\,\gamma^2}} x\,e^u du \\ &= \frac{\alpha}{\beta}\left[e^{\frac{t^2\,\beta}{2\,\gamma^2}} - 1\right] \end{aligned} \tag{4.8}$$

4.2.2 Odds function

This is the odds of the probability that the failure of a unit is bound to happen at a given time, to the probability that it is bound to survive beyond that time. That is;

$$\begin{aligned} O(x) &= \frac{F(x)}{S(x)} \\ &= \frac{1 - e^{\frac{\alpha}{\beta}(1 - e^{\frac{\beta x^2}{2\gamma^2}})}}{e^{\frac{\alpha}{\beta}(1 - e^{\frac{\beta x^2}{2\gamma^2}})}} \end{aligned} \tag{4.9}$$

4.3 Moment and moment generating function

4.3.1 r^{th} non-central moment

This is an important property of any distribution and used in obtaining some measures like shapes, dispersion, central tendencies and so on.

Suppose a random variable X follows GomR distribution, the r^{th} non-central moment, μ'_r, is obtained using the expression

$$\begin{aligned} \mu'_r &= E(X^r) \\ &= \int_0^\infty x^r f(x) dx \\ &= \int_0^\infty x^r \frac{\alpha x}{\gamma^2} e^{\frac{\beta x^2}{2\gamma^2}} e^{\frac{\alpha}{\beta}[1 - e^{\frac{x^2\beta}{2\gamma^2}}]} dx \end{aligned}$$

Recalling the useful representation (2.4)

$$\begin{aligned} \mu'_r &= \int_0^\infty x^r \frac{\alpha x}{\gamma^2} \sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty} \frac{1}{b_1!}(-1)^{b_2}\left(\frac{\alpha}{\beta}\right)^{b_1}\binom{b_1}{b_2} e^{\frac{x^2\beta(1+b_2)}{2\gamma^2}} dx \\ &= \frac{\alpha}{\gamma^2}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty} \mathcal{B}_{b_1 b_2}\int_0^\infty x^{r+1} e^{\frac{x^2\beta(1+b_2)}{2\gamma^2}} dx \\ where \quad \mathcal{B}_{b_1 b_2} &= \frac{1}{b_1!}(-1)^{b_2}\left(\frac{\alpha}{\beta}\right)^{b_1}\binom{b_1}{b_2} \end{aligned}$$

Now

$$\begin{aligned} \mu'_r &= \frac{\alpha}{\gamma^2}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty} \mathcal{B}_{b_1 b_2}\int_0^\infty x^{r+1} e^{\frac{-x^2\beta(-1-b_2)}{2\gamma^2}} dx \\ let \quad m &= \frac{x^2\beta(-1-b_2)}{2\gamma^2} \qquad and \qquad x = \frac{(2m)^{\frac{1}{2}}\gamma}{(\beta(-1-b_2))^{\frac{1}{2}}} \\ then \quad \frac{dm}{dx} &= \frac{\beta x(-1-b_2)}{\gamma^2} \qquad and \qquad dx = \frac{\gamma^2 dm}{(\beta x(-1-b_2))} \end{aligned}$$

Now

$$\begin{aligned}
\mu'_r &= \frac{\alpha}{\gamma^2}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\mathcal{B}_{b_1b_2}\frac{1}{\beta(-1-b_2)}\int_0^{\infty}\left(\frac{(2\,m)^{\frac{1}{2}}\gamma}{(\beta(-1-b_2))^{\frac{1}{2}}}\right)^{r+1}e^{-m}\,.\,\frac{dm}{x}\\
&= \frac{\alpha}{\gamma^2}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\mathcal{B}_{b_1b_2}\frac{1}{\beta(-1-b_2)}\int_0^{\infty}\left(\frac{(2\,m)^{\frac{1}{2}}\gamma}{(\beta(-1-b_2))^{\frac{1}{2}}}\right)^{r}e^{-m}\,.\,dm\\
&= \frac{\alpha}{\gamma^2}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\frac{\mathcal{B}_{b_1b_2}\,2^{\frac{r}{2}}\,\gamma^r}{(\beta(-1-b_2))^{1+\frac{r}{2}}}\int_0^{\infty}m^{\frac{r}{2}}\,e^{-m}\,.\,dm\\
&= \frac{\alpha}{\gamma^2}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\frac{\mathcal{B}_{b_1b_2}\,2^{\frac{r}{2}}\,\gamma^r}{(\beta(-1-b_2))^{1+\frac{r}{2}}}\,.\,\Gamma(1+\frac{r}{2})
\end{aligned}$$

therefore

$$\mu'_r = \frac{\alpha}{\gamma^2}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\frac{\mathcal{B}_{b_1b_2}\,2^{\frac{r}{2}}\,\gamma^r}{(\beta(-1-b_2))^{1+\frac{r}{2}}}\,.\,\Gamma(1+\frac{r}{2}) \tag{4.10}$$

4.3.2 MGF

Generally, the MGF of any random variable can be obtained using the relation

$$M(\theta) = E(e^{\theta x})$$

since X is a continuous random variable with pdf f(x),

$$M(\theta) = \int_0^{\infty} e^{\theta x} f(x)dx$$

or in simpler form

$$\begin{aligned}
M(\theta) &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^r\,f(x)dx\\
&= \sum_{r=0}^{\infty}\frac{t^r}{r!}\,\mu'_r\\
since\quad e^{tx} &= \sum_{r=0}^{\infty}\frac{(tx)^r}{r!}\\
and\quad \mu'_r\ &is\ the\ r^{th} non-central\ moment
\end{aligned}$$

the MGF of GomR is

$$\frac{\alpha}{\gamma^2}\sum_{r=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\frac{t^r}{r!}\frac{\mathcal{B}_{b_1b_2}\,2^{\frac{r}{2}}\,\gamma^r}{(\beta(-1-b_2))^{1+\frac{r}{2}}}\,.\,\Gamma(1+\frac{r}{2}) \tag{4.11}$$

4.4 Mean and variance of GomR distribution

These are obtained from the r^{th} non-central moment of GomR distribution.

4.4.1 Mean

If in (4.10), $r = 1$, the resulting equation is the mean (1^{st} moment) of GomR distribution, That is,

$$\begin{aligned} \mu_1' &= E(X) \\ &= \frac{\alpha}{\gamma^2} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \frac{\mathcal{B}_{b_1 b_2}\, 2^{\frac{r}{2}}\, \gamma}{(\beta(-1 - b_2))^{\frac{3}{2}}} \cdot \Gamma(\frac{3}{2}) \\ &= \frac{\alpha}{\gamma^2} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \frac{0.8862\, \mathcal{B}_{b_1 b_2}\, \gamma}{(\beta(-1 - b_2))^{\frac{3}{2}}} \end{aligned} \quad (4.12)$$

4.4.2 Variance

Using the relation

$$Var(X) = E(X^2) - [E(X)]^2$$

where $E(X^2)$ is the 2^{nd} moment and obtained when $r = 2$ in (4.10).
Hence

$$\begin{aligned} \mu_2' &= E(X^2) \\ &= \frac{\alpha}{\gamma^2} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \frac{\mathcal{B}_{b_1 b_2}\, 2^2\, \gamma^2}{(\beta(-1 - b_2))^2} \cdot \Gamma(2) \\ &= \frac{\alpha}{\gamma^2} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \frac{\mathcal{B}_{b_1 b_2}\, \gamma^2}{(\beta(-1 - b_2))^2} \end{aligned} \quad (4.13)$$

therefore the variance of GomR distribution is

$$Var(X) = \frac{\alpha}{\gamma^2} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \frac{\mathcal{B}_{b_1 b_2}\, \gamma^2}{(\beta(-1 - b_2))^2} - \left(\frac{\alpha}{\gamma^2} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \frac{0.8862\, \mathcal{B}_{b_1 b_2}\, \gamma}{(\beta(-1 - b_2))^{\frac{3}{2}}} \right)^2 \quad (4.14)$$

4.5 Entropy

Renyi entropy by [17] a measure of uncertainty was defined as

$$I_R(c) = \frac{1}{1-c} \log \int_0^{\infty} f^c(x)dx \qquad c > 0 \;, c \neq 1 \quad (4.15)$$

Suppose a random variable X follows the GomR distribution, the degree of uncertainty is obtained as follows

$$\begin{aligned} f^c(x) &= \left(\frac{\alpha x}{\gamma^2}\, e^{\frac{\beta x^2}{2\gamma^2}}\, e^{\frac{\alpha}{\beta}[1 - e^{\frac{x^2 \beta}{2\gamma^2}}]} \right)^c \\ & \left(\frac{\alpha}{\gamma^2} \right)^c x^c\, e^{\frac{\beta x^2\, c}{2\gamma^2}}\, e^{\frac{\alpha\, c}{\beta}[1 - e^{\frac{x^2 \beta}{2\gamma^2}}]} \end{aligned}$$

using the expansions earlier

$$e^{\frac{\alpha\, c}{\beta}[1 - e^{\frac{x^2 \beta}{2\gamma^2}}]} = \sum_{b_4=0}^{\infty} \sum_{b_5=0}^{\infty} \frac{1}{b_3!} \left(\frac{\alpha\, c}{\beta} \right)^{b_4} \binom{b_4}{b_5} (-1)^{b_5}\, e^{\frac{x^2 \beta\, b_5}{2\gamma^2}}$$

implying that

$$
\begin{aligned}
f^c(x) &= \left(\frac{\alpha}{\gamma^2}\right)^c x^c \sum_{b_4=0}^{\infty}\sum_{b_5=0}^{\infty} \frac{1}{b_3!}\left(\frac{\alpha\, c}{\beta}\right)^{b_4}\binom{b_4}{b_5}(-1)^{b_5}\, e^{\frac{x^2\beta\,(c+b_5)}{2\gamma^2}} \\
&= \left(\frac{\alpha}{\gamma^2}\right)^c x^c \sum_{b_4=0}^{\infty}\sum_{b_5=0}^{\infty} \mathcal{C}_{b4,b_5}\, e^{\frac{x^2\beta\,(c+b_5)}{2\gamma^2}} \\
&\quad where \quad \mathcal{C}_{b4,b_5} = \sum_{b_4=0}^{\infty}\sum_{b_5=0}^{\infty} \frac{1}{b_3!}\left(\frac{\alpha\, c}{\beta}\right)^{b_4}\binom{b_4}{b_5}(-1)^{b_5}
\end{aligned}
$$

now

$$
I_R(c) = \frac{1}{1-c}\log\left(\left(\frac{\alpha}{\gamma^2}\right)^c \sum_{b_4=0}^{\infty}\sum_{b_5=0}^{\infty}\mathcal{C}_{b4,b_5}\int_0^{\infty} x^c\, e^{\frac{x^2\beta\,(c+b_5)}{2\gamma^2}}\,dx\right) \qquad c>0\ ,c\neq
$$

$$
let \quad n = e^{\frac{-\,x^2\beta\,(-c\,-\,b_5)}{2\gamma^2}} \qquad then \qquad \frac{dn}{dx} = \frac{x\,\beta\,(-c\,-\,b_5)}{\gamma^2} \qquad and \qquad x = \frac{(2n)^{\frac{1}{2}}\gamma}{(\beta\,(-c\,-\,b_5))^{\frac{1}{2}}}
$$

$$
\begin{aligned}
&= \frac{1}{1-c}\log\left[\frac{\alpha^c}{\gamma^{2c}}\sum_{b_4=0}^{\infty}\sum_{b_5=0}^{\infty}\mathcal{C}_{b4,b_5}\int_0^{\infty}\left(\frac{(2n)^{\frac{1}{2}}\gamma}{(\beta\,(-c\,-\,b_5))^{\frac{1}{2}}}\right)^c e^{-n}\,\frac{\gamma^2\,dn}{\beta\,x\,(-c\,-\,b_5)}\right] \\
&= \frac{1}{1-c}\log\left[\frac{\alpha^c}{\gamma^{2c-2}}\sum_{b_4=0}^{\infty}\sum_{b_5=0}^{\infty}\mathcal{C}_{b4,b_5}\int_0^{\infty}\left(\frac{(2n)^{\frac{1}{2}}\gamma}{(\beta\,(-c\,-\,b_5))^{\frac{1}{2}}}\right)^{c-1} e^{-n}\,\frac{\gamma^2\,dn}{\beta\,(-c\,-\,b_5)}\right] \\
&= \frac{1}{1-c}\log\left[\frac{\alpha^c\,\gamma^{1-c}\,2^{\frac{c-1}{2}}}{(\beta\,(-c\,-\,b_5))^{\frac{c-3}{2}}}\sum_{b_4=0}^{\infty}\sum_{b_5=0}^{\infty}\mathcal{C}_{b4,b_5}\int_0^{\infty} n^{\frac{c-1}{2}}\,e^{-n}\,dn\right]
\end{aligned}
$$

$$
= \frac{1}{1-c}\log\left[\frac{\alpha^c\,\gamma^{1-c}\,2^{\frac{c-1}{2}}}{(\beta\,(-c\,-\,b_5))^{\frac{c-3}{2}}}\sum_{b_4=0}^{\infty}\sum_{b_5=0}^{\infty}\mathcal{C}_{b4,b_5}\,\Gamma(\frac{1+c}{2})\right] \tag{4.16}
$$

5 Parameter Estimation

This section provides the estimates of the three unknown parameters (α, β, γ) of GomR distribution using the method of Maximum Product of Spacing in addition to the MLE from an earlier study.

5.1 MLE

Assume $X_1, X_2, \ldots, X_n$ to be a random sample of size n following GomR distribution. The likelihood function is

$$\begin{aligned} L &= \prod_{i=1}^{n} f(x_i;\varepsilon) \\ &= \prod_{i=1}^{n}\left\{\frac{\alpha x_i}{\gamma^2}\, e^{\frac{\beta x_i^2}{2\gamma^2}}\, e^{\frac{\alpha}{\beta}[1-e^{\frac{x_i^2\beta}{2\gamma^2}}]}\right\} \\ &= \left(\frac{\alpha}{\gamma^2}\right)^n \sum_{i=1}^{n} x_i \prod_{i=1}^{n}\left\{e^{\frac{\beta\, x_i^2}{2\,\gamma^2}+\frac{\alpha}{\beta}(1-e^{\frac{\beta\, x_i^2}{2\,\gamma^2}})}\right\} \end{aligned}$$

The log-likelihood function

$$L(\theta) = n\log\alpha - 2n\log\gamma + \sum_{i=1}^{n}\log(x_i) + \frac{\beta}{2\,\gamma^2}\sum_{i=1}^{n} x_i^2 + \frac{\alpha}{\beta}\sum_{i=1}^{n}\left[1 - e^{\frac{\beta\, x_i^2}{2\,\gamma}}\right]$$

To obtain the estimates of the parameters $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$, we differentiate $L(\theta)$ wrt individual parameter and equate to zero. The resulting differentials are,

$$\frac{\partial L(\theta)}{\partial\alpha} = \frac{n}{\alpha} + \frac{1}{\beta}\sum_{i=1}^{n}\left[1 - e^{\frac{\beta\, x_i^2}{2\,\gamma^2}}\right] \tag{5.1}$$

$$\frac{\partial L(\theta)}{\partial\beta} = \frac{1}{2\,\gamma^2}\cdot\sum_{i=1}^{n} x_i^2 - \frac{\alpha}{2\beta\gamma^2}\cdot\sum_{i=1}^{n}\left(x_i^2\, e^{\frac{\beta x_i^2}{2\gamma^2}}\right) - \frac{\alpha}{\beta^2}\sum_{i=1}^{n}\left(1 - e^{\frac{\beta x_i^2}{2\gamma^2}}\right) \tag{5.2}$$

$$\frac{\partial L(\theta)}{\partial\gamma} = \frac{2n}{\gamma} - \frac{\beta}{\gamma^3}\sum_{i=1}^{n} x_i^2 - \frac{\alpha}{\gamma^3}\sum_{i=1}^{n}\left(x_i^2\, e^{\frac{\beta x_i^2}{2\gamma^2}}\right) \tag{5.3}$$

The above equations are not in explicit form, hence, do not have exact solution. For this reason, we use the Newton-Raphson method of iteration, to obtain the MLEs of the equations analytically.

To enable the construction of confidence intervals of the parameters and hypothesis testing of the GomR distribution. The elements of the observed information matrix are found as follows.

$$\frac{\partial L(\theta)}{\partial\alpha^2} = \frac{-n}{\alpha^2} \tag{5.4}$$

$$\frac{\partial L(\theta)}{\partial\beta^2} = \frac{-\alpha x^4}{4\beta\gamma^4}\sum_{i=1}^{n}\left(e^{\frac{\beta x^2}{2\gamma^2}}\right) + \frac{\alpha x^2}{\gamma^2\beta^2}\sum_{i=1}^{n}\left(e^{\frac{\beta x^2}{2\gamma^2}}\right) + \frac{\alpha}{\beta^3}\sum_{i=1}^{n}\left(1 - e^{\frac{\beta x^2}{2\gamma 62}}\right) \tag{5.5}$$

$$\frac{\partial L(\theta)}{\partial\gamma^2} = \frac{\beta}{\gamma^4}\sum_{i=1}^{n} x_i^2 - \frac{2n}{\gamma^4} - \frac{\alpha\beta}{\gamma^6}\sum_{i=1}^{n} x_i^4 e^{\frac{\beta x^2}{2\gamma^2}} - \frac{\alpha}{\gamma^4}\sum_{i=1}^{n} x_i^2 e^{\frac{\beta x^2}{2\gamma^2}} \tag{5.6}$$

$$\frac{\partial L(\theta)}{\partial\alpha\partial\beta} = \frac{-1}{2\beta\gamma^2}\sum_{i=1}^{n} x_i^2 e^{\frac{\beta x^2}{2\gamma 62}} - \frac{1}{\beta^2}\sum_{i=1}^{n}\left[1 - e^{\frac{\beta x^2}{2\gamma 62}}\right] \tag{5.7}$$

$$\frac{\partial L(\theta)}{\partial\alpha\partial\gamma} = \frac{1}{\gamma^3}\sum_{i=1}^{n} x_i^2 e^{\frac{\beta x^2}{2\gamma 62}} \tag{5.8}$$

$$\frac{\partial L(\theta)}{\partial\beta\partial\gamma} = \frac{-1}{\gamma^3}\sum_{i=1}^{n} x_i^2 + \frac{\alpha}{2\gamma^5}\sum_{i=1}^{n} x_i^4 e^{\frac{\beta x^2}{2\gamma 62}} \tag{5.9}$$

5.2 MPS

The maximum likelihood estimation is the most common and widely used method but in cases such as that involving compound continuous distributions and large samples, the method may break down.

[12] introduced the MPS method serving as an alternative to ML method, a powerful one. Also, [13] independently studied the method as an approximation to Kullback-Leibler information and explained its consistency property.

If $X_1, X_2, \ldots, X_n$ are random samples from GomR distribution having CDF $F(x, \varepsilon)$ and $X_{(1)} \leq X_{(2)}, \ldots X_{(n)}$ represents the corresponding ordered sample. The spacing

$$\mho_i = F(x_{(i)}) \; - \; F(x_{(i-1)}) \qquad for \quad i = 1, 2, \ldots, n+1$$

where

$$F(x_{(0)}) \; = \; 0 \qquad and \qquad F(x_{(n+1)}) = 1$$

Since we are sampling from GomR distribution,

$$F(x_{(i)}) \; = \; 1 \; - \; e^{\frac{\alpha}{\beta}\left(1 - e^{\frac{\beta x_{(i)}^2}{2\gamma 62}}\right)} \tag{5.10}$$

and

$$F(x_{(i-1)}) \; = \; 1 \; - \; e^{\frac{\alpha}{\beta}\left(1 - e^{\frac{\beta x_{(i-1)}^2}{2\gamma 62}}\right)} \tag{5.11}$$

then

$$\mho_i \; = \; \left(1 \; - \; e^{\frac{\alpha}{\beta}\left(1 - e^{\frac{\beta x_{(i)}^2}{2\gamma 62}}\right)}\right) - \left(1 \; - \; e^{\frac{\alpha}{\beta}\left(1 - e^{\frac{\beta x_{(i-1)}^2}{2\gamma 62}}\right)}\right) \tag{5.12}$$

The parameter estimates are obtained by maximizing

$$T \; = \; \frac{1}{n+1} \sum_{i=1}^{n+1} \log_e \mho_i \tag{5.13}$$

$$T \; = \; \frac{1}{n+1} \sum_{i=1}^{n+1} \log_e \left\{ e^{\frac{\alpha}{\beta}\left(1 - e^{\frac{\beta x_{(i)}^2}{2\gamma 62}}\right)} \; - \; e^{\frac{\alpha}{\beta}\left(1 - e^{\frac{\beta x_{(i-1)}^2}{2\gamma 62}}\right)} \right\} \tag{5.14}$$

The parameters estimates $\hat{\alpha}_{MPS}, \hat{\beta}_{MPS}, \hat{\gamma}_{MPS}$ can be found by differentiating T wrt individual parameters and solving the non-linear equations

$$\frac{\partial T(\alpha, \beta, \gamma)}{\partial \alpha} \; = \; \frac{1}{n+1} \cdot \sum_{i=1}^{n+1} \frac{1}{\mho_i(\alpha, \beta, \gamma)} \cdot \left[\mho_1(x_{(i)}.\varepsilon) \; - \; \mho_1(x_{(i-1)}.\varepsilon)\right] \tag{5.15}$$

$$\frac{\partial T(\alpha, \beta, \gamma)}{\partial \beta} \; = \; \frac{1}{n+1} \cdot \sum_{i=1}^{n+1} \frac{1}{\mho_i(\alpha, \beta, \gamma)} \cdot \left[\mho_2(x_{(i)}.\varepsilon) \; - \; \mho_2(x_{(i-1)}.\varepsilon)\right] \tag{5.16}$$

$$\frac{\partial T(\alpha, \beta, \gamma)}{\partial \gamma} \; = \; \frac{1}{n+1} \cdot \sum_{i=1}^{n+1} \frac{1}{\mho_i(\alpha, \beta, \gamma)} \cdot \left[\mho_3(x_{(i)}.\varepsilon) \; - \; \mho_3(x_{(i-1)}.\varepsilon)\right] \tag{5.17}$$

where

$$\mho_1(x_{(i-1)},\varepsilon) = e^{\frac{\alpha}{\beta}(1-e^{\frac{\beta x_{(i-1)}}{2\gamma^2}})}.\frac{1}{\beta}\left(1 - e^{\frac{\beta x_{(i-1)}}{2\gamma^2}}\right) \tag{5.18}$$

$$\mho_1(x_{(i)},\varepsilon) = e^{\frac{\alpha}{\beta}(1-e^{\frac{\beta x_{(i)}}{2\gamma^2}})}.\frac{1}{\beta}\left(1 - e^{\frac{\beta x_{(i)}}{2\gamma^2}}\right) \tag{5.19}$$

$$\begin{aligned}\mho_2(x_{(i-1)},\varepsilon) &= e^{\frac{\alpha}{\beta}(1-e^{\frac{\beta x_{(i-1)}}{2\gamma^2}})}.\left\{\left(1 - e^{\frac{\beta x_{(i-1)}}{2\gamma^2}}\right).\frac{-\alpha}{\beta^2} + \frac{\alpha}{\beta}. - e^{\frac{\beta x_{(i-1)}}{2\gamma^2}}.\frac{x}{2\gamma^2}\right\}\\ &= e^{\frac{\alpha}{\beta}(1-e^{\frac{\beta x_{(i-1)}}{2\gamma^2}})}.\frac{-\alpha}{\beta}\left\{\frac{\left(1 - e^{\frac{\beta x_{(i-1)}}{2\gamma^2}}\right)}{\beta} - \frac{x_{((i-1))}.e^{\frac{\beta x_{(i-1)}}{2\gamma^2}}}{2\gamma^2}\right\}\end{aligned} \tag{5.20}$$

$$\begin{aligned}\mho_3(x_{(i-1)},\varepsilon) &= e^{\frac{\alpha}{\beta}(1-e^{\frac{\beta x_{(i-1)}}{2\gamma^2}})}.\frac{\alpha}{\beta}. - e^{\frac{\beta x_{(i-1)}}{2\gamma^2}}.\frac{\beta x_{(i-1)}}{\gamma^3}\\ &= \frac{\alpha x_{(i-1)}}{\gamma^3}.e^{\frac{\alpha}{\beta}(1-e^{\frac{\beta x_{(i-1)}}{2\gamma^2}})+\frac{\beta x_{(i-1)}}{2\gamma^2}}\end{aligned} \tag{5.21}$$

The solutions to equations (5.15),(5.16) and (5.17) are the MPS parameter estimates. However, the equations cannot be obtained analytically but rather with the use of numerical solutions.

6 Applications

We portray the advantage of GomR distribution over some related distributions having at least two parameters in fitting two rainfall data sets. The comparison was done using the log-likelihood, Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Corrected Akaike's Information Criteria (CAIC) and Hannan-Quinn Information Criteria (HQIC).

$$\begin{aligned}\text{AIC} &= 2(\text{ll}) + 2\text{k}\\ \text{BIC} &= -(2 * \text{ll}) + (\text{k} * (\log(n)))\\ \text{CAIC} &= -2(\text{ll}) + 2\text{k}(\text{k}+1)/(\text{n}-\text{k}-1)\\ \text{HQIC} &= -(2(\text{ll})) + (2 * \text{k} * \log(\log(\text{n})))\end{aligned}$$

where ll is the log-likelihood, n is the sample size and k is the number of parameters to be fitted. These information criteria will serve as scores for selecting the distribution that best fit the data. Using rainfall data sets from three different regions, the goodness of fit of the GomR distribution was compared with Generalized Weibull Rayleigh (GWR) by [18], Exponentiated Weibull Rayleigh (EWR) by [19], Type (II) Topp Leone Generalized Inverse Rayleigh (TIITLGIR) by [20], Kumarawamy Exponential Inverse Raylrigh (KEIR) by [21], Negative Binomial Marshall-Olkin Rayleigh (NBMOR) by [22] and Exponentiated Weibull (EW) by [6].

6.1 Malaysian Rainfall data

These consists of 30 years means of maximum daily rainfall from 1975-2004 at 35 stations in the middle and west of peninsular Malaysia. Table below provide the descriptive statistics of the data Table 3 presents each distribution with their maximum likelihood estimates and maximum product of spacing estimates while Table 4 the distributions and their corresponding measures of comparison.

Table 1. Malaysian rainfall data

1.134	1.196	1.181	1.178	1.048	1.077	0.835	1.163	0.880
1.056	1.164	0.914	1.141	1.068	1.007	1.027	1.298	0.842
0.991	0.955	0.703	0.953	1.018	1.003	1.106	1.110	1.249
1.092	1.187	1.047	0.989	0.955	1.234	0.937	0.933	

Table 2. Descriptive statistics of Malaysian rainfall data

Variables	Description
Sample size	35
Maximum value, Minimum value	1.298, 0.703
Mode	1.05, 1.15
Kurtosis, Skewness	-0.0760, -0.3504
Mean, Median, Variance	1.0477, 1.048, 0.0172

Table 3. Models parameter estimates

	MLEs				MPSs			
Distribution	α	β	γ	θ	α	β	γ	θ
GomR	0.0719	8.5914	1.0561	—-	0.0949	7.4284	1.0303	—-
GWR	40.8144	1.9501	—-	—-	28.6915	1.8642	—-	—-
EWR	0.4924	1.5769	1.0194	3.1147	0.2412	1.2453	1.4860	2.6946
TIITLGIR	7.8861	0.0418	79.3689	—-	6.1335	0.0623	50.5646	—-
KEIR	4.4906	79.3503	18.9096	0.0613	12.5434	50.5068	0.9696	0.3852
NBMOR	494.5516	1.3154	0.2903	—-	1.2558	8.8050	0.5167	—-
EW	1.4725	0.9432	7.6228	—-	1.3744	0.9367	7.1843	—-

Table 4. Log-likelihood and information criteria

	MLEs					MPSs				
Distribution	AIC	BIC	CAIC	HQIC	ll	AIC	BIC	CAIC	HQIC	ll
GomR	35.2638	30.5977	48.0380	48.8745	-20.6319	46.669	51.33504	47.44319	48.27972	-20.3345
GWR	35.4114	32.30069	43.7864	44.4852	-19.7057	42.8132	45.9239	43.1882	43.88701	-19.4066
EWR	36.8744	30.6530	54.2077	55.0220	-22.4372	52.14272	58.36411	53.4761	54.2903	-22.0714
TIITLGIR	37.6353	32.9692	50.4095	51.2460	-21.8176	49.0422	53.7082	49.8164	50.65292	-21.5211
KEIR	35.6353	29.4139	52.9686	53.7829	-21.8176	51.0392	57.2606	52.3725	53.1868	-21.5196
NBMOR	37.4214	32.7554	50.1956	51.0321	-21.7107	14.0908	18.7568	14.86495	15.70148	-4.04538
EW	38.8561	34.1900	51.6302	52.4668	-22.4280	50.2632	54.9292	51.0374	51.8739	-22.1316

6.2 Nigerian rainfall data

These consists of 115 years average annual rainfall in Nigeria from 1901-2015. Below is the descriptive statistics of the data Table 7 gives each distribution with their maximum likelihood estimates and maximum product of spacing estimates while Table 8 the distributions and their corresponding measures of comparison.

6.3 Argentine rainfall data

These consist of 25 years annual rainfall of Argentina from 1991-2015 . Table below provide the descriptive statistics of the data Table 3 presents each distribution with their maximum likelihood estimates and maximum product of spacing estimates while Table 4 the distributions and their corresponding measures of comparison.

Table 5. Nigerian rainfall data

1.0158	1.0103	0.9736	0.9584	1.1158	1.0478	0.8551	1.0077	1.1099
0.9432	1.0257	0.8447	0.8696	0.7831	0.9807	1.0687	0.9952	1.0350
0.8836	1.0099	1.0079	0.9125	0.9785	1.0838	1.0005	0.9254	1.0801
1.0457	1.0189	1.0391	1.0490	0.9110	1.0050	1.0240	0.9847	0.9921
0.9343	0.9407	1.0718	0.9947	0.9875	0.9195	0.9584	0.8958	0.9354
0.9492	0.9964	0.9839	0.9626	0.9022	1.0125	1.0204	0.9846	1.1070
1.0952	0.8952	1.1184	0.9070	1.0138	1.0551	0.9284	1.0697	1.1027
0.9794	0.9674	1.0042	0.9595	1.0343	1.0256	0.9325	0.9087	0.8770
0.8120	0.9448	0.9642	0.9146	0.8699	1.0143	0.9688	1.0006	0.9157
0.8568	0.7301	0.8905	0.8858	0.8840	0.828	0.9479	0.9160	0.8901
0.9751	0.9086	0.9576	0.9765	0.9542	1.0192	0.9967	0.9622	1.0135
0.9371	0.8835	0.9046	1.0267	0.9644	0.8712	0.9599	1.0104	1.0931
0.9428	0.9497	0.8024	1.0183	0.7660	0.9070	0.8039		

Table 6. Descriptive statistics of Nigerian rainfall data

Variables	Description
Sample size	115
Maximum value, Minimum value	1.1184 , 0.7302
Mode	1.025
Kurtosis, Skewness	0.1403, -0.3703
Mean, Median, Variance	0.9640, 0.9674, 0.0061

Table 7. Models parameters estimates

	MLEs				MPSs			
distribution	α	β	γ	θ	α	β	γ	θ
GomR	0.0074	9.4394	0.8154		0.0092	9.5525	0.8334	
GWR	304.4545	2.5786			242.9539	2.5300		
EWR	0.2313	2.0739	1.3902	3.8271	0.0806	0.7653	4.5070	5.0639
TIITLGIR	0.2889	40.7968	988.3569		0.3505	26.7564	776.7488	
KEIR	90.8850	989.3505	0.3708	0.2021	0.2919	313.5956	0.2266	85.9585
NBMOR	494.9276	1.3153	0.2903		1.5195	1.5512	0.8155	
EW	2.1413	1.0592	9.6423		1.9931	1.0539	9.6436	

Table 8. Log-likelihood and information criteria

	MLEs					MPSE				
Distribution	AIC	BIC	CAIC	HQIC	ll	AIC	BIC	CAIC	HQIC	ll
GomR	246.1634	237.9286	258.3796	261.5059	-126.0817	245.9038	237.669	245.6876	242.5613	-125.9519
GWR	237.568	232.0781	245.6751	247.7963	-120.784	237.1504	231.6605	237.0433	234.9221	-120.5752
EWR	256.3186	245.3389	255.955	251.862	-132.1593	256.019	245.0393	255.6554	251.5624	-132.0095
TIITLGIR	258.1548	249.92	257.9386	254.8123	-132.0774	257.8956	249.6608	257.6794	254.5531	-131.9478
KEIR	256.1548	245.175	272.5184	276.6114	-132.0774	251.6908	240.7111	251.3272	247.2342	-129.8454
NBMOR	254.1214	245.8866	253.9052	250.7789	-130.0607	85.5883	77.3535	85.37208	82.24584	-45.7942
EW	258.2878	250.053	258.0716	254.9453	-132.1439	258.0172	249.7824	257.801	254.6747	-132.0086

From Tables 4 and 8, except for NBMOR and GWR in MPS, GomR had the lowest information criteria and highest log-likelihood than the distributions compared with. These implies that although GWR, EWR, TIITGIR, KEIR, NBMOR, EW are good distributionss for the Malaysian and

Table 9. Argentine rainfall data

6.163	6.775	5.685	5.607	4.687	5.871	6.149	5.809
5.936	6.673	6.675	7.040	5.500	5.375	5.277	5.508
5.447	4.944	4.972	5.155	5.178	5.794	5.427	6.523
5.988							

Table 10. Descriptive statistics of Argentine rainfall data

Variables	Description
Sample size	25
Maximum value, Minimum value	7.04, 4.687
Mode	5.75
Kurtosis, Skewness	-0.65405, 0.3784
Mean, Median, Variance	5.7663, 5.685, 0.3844

Table 11. Models parameter estimates

	MLEs				MPSs			
Distribution	α	β	γ	θ	α	β	γ	θ
GomR	0.0175	2.2313	2.9276	—	0.0262	2.0488	2.9660	—
GWR	174.1254	0.4131	—	—	94.4851	0.3890	—	—
EWR	—	—	—	—	—	—	—	—
TIITLGIR	14.4895	0.4169	132.4804	—	15.8917	0.3108	77.9342	—
KEIR	19.8197	132.5918	0.2707	32.6324	21.8488	77.9643	0.6062	11.8540
NBMOR	—	—	—	—	—	—	—	—
EW	76.0733	0.3380	2.3593	—	6.3212	0.2116	4.1536	—

Table 12. Log-likelihood and information criteria

	MLEs					MPSs				
Distribution	AIC	BIC	CAIC	HQIC	ll	AIC	BIC	CAIC	HQIC	ll
GomR	47.4706	43.8140	46.3277	46.4564	-26.7353	48.0772	44.4206	46.9343	47.0630	-27.0386
GWR	41.019	38.5813	40.4736	40.3429	-22.5095	41.6588	39.2211	41.1134	40.9827	-22.8294
EWR	—	—	—	—	—	—	—	—	—	—
TIITLGIR	40.2688	36.6122	39.1259	39.2546	-23.1344	40.8322	37.1756	39.6893	39.8180	-23.4161
KEIR	38.2688	33.3933	36.2688	36.9165	-23.1344	38.8312	33.9557	36.8312	37.4789	-23.4156
NBMOR	—	—	—	—	—	—	—	—	—	—
EW	38.9922	35.3356	37.8493	37.9780	-22.4961	40.155	36.4984	39.0121	39.14081	-23.0775

Nigerian rainfall data, the GomR distribution provides a better fit considering MLE while GomR only provides a better fit than EWR, TIITLGIR, KEIR and EW distributions when MPSE was considered. However, Table 12 showed otherwise where GomR distribution was the worst in fitting Argentine rainfall data while NBMOR and GWR distributions could not fit the data at all.

7 Simulation Study

In this subsection, Monte Carlo approach to simulation study was adopted. The important objective of simulations is determine the most efficient between MLE and MPS methods for the GomR distribution parameters. Using different parameters values and sample sizes (30 − 1000), the estimation methods were compared based on bias and root mean square error (RMSE) of the estimates.

$$\text{Bias} = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i - \theta_i) \tag{7.1}$$

$$\text{RMSE} = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_i - \theta_i)^2} \tag{7.2}$$

Steps adopted are as follows:

1. Set the sample size and the vector of parameter values $\theta = (\alpha, \beta, \gamma)$
2. Generate sample of size n from GomR distribution
3. Using the values obtained above, obtain $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ using MLE and MPSE.
4. In 1000 times, repeat steps (2) and (3).
5. Using θ and $\hat{\theta}$, compute Bias and RMSE.

Table 13. Means Bias and RMSE for the parameter estimates when $\alpha = 2.3, \beta = 5.2, \gamma = 1.0$

	MLEs			MPSE		
n	Means	Bias	RMSE	Means	Bias	RMSE
30	2.2712	-0.0288	0.9399	2.7118	0.4118	1.0571
	3.8501	0.3501	1.0975	3.2236	-0.2764	1.1148
	0.9942	-0.0058	0.0714	1.0259	0.0259	0.0717
100	2.3782	0.0782	0.5821	2.5866	0.2866	0.6718
	5.4703	0.2703	0.7745	5.1854	-0.0146	0.7830
	1.0140	0.0140	0.0405	1.0286	0.0286	0.0475
250	2.3629	0.0629	0.3471	2.4672	0.1672	0.4195
	5.4166	0.2166	0.5307	5.2666	0.0666	0.4760
	1.0148	0.0148	0.0262	1.0215	0.0215	0.0343
350	2.3672	0.0672	0.2924	2.4479	0.1479	0.3462
	5.3862	0.1868	0.4378	5.2728	0.0728	0.3928
	1.0146	0.0146	0.0233	1.0199	0.0199	0.0296
1000	2.3620	0.0620	0.1911	2.3980	0.0980	0.2136
	5.3243	0.1243	0.2696	5.2759	0.0759	0.2330
	1.0117	0.0117	0.0181	1.0141	0.0141	0.0204

Table 14. Means, Bias and RMSEs for the parameter estimates when $\alpha = 1.8, \beta = 5.9, \gamma = 0.5$

	MLEs			MPSs		
n	Means	Bias	RMSE	Means	Bias	RMSE
30	1.8363	0.0363	0.8320	2.2900	0.4900	1.1376
	6.3487	0.4487	1.1556	5.7491	-0.1509	1.1660
	0.5013	0.0013	0.0326	0.5213	0.0213	0.0432
100	1.8655	0.0655	0.04860	2.0446	0.2446	0.5948
	6.1721	0.2721	0.7229	5.9275	0.0275	0.6674
	0.5867	0.0067	0.0193	0.5146	0.0146	0.0250
250	1.8546	0.0546	0.3145	1.9402	0.1402	0.3537
	6.1300	0.2300	0.4576	6.0071	0.1071	0.4181
	0.5075	0.0075	0.0139	0.5112	0.0112	0.0167
350	1.8533	0.0533	0.2538	1.9215	0.1215	0.2915
	6.0919	0.1919	0.3948	6.0008	0.1008	0.3375
	0.5069	0.0069	0.0118	0.5100	0.0100	0.0146
1000	1.8519	0.0519	0.1612	1.8822	0.0822	0.1793
	6.0447	0.1447	0.2544	6.0073	0.1073	0.2225
	0.5061	0.0061	0.0092	0.5075	0.0075	0.0104

Table 15. Means, Bias and RMSEs for the parameter estimates when $\alpha = 1.2, \beta = 6.5, \gamma = 0.2$

	MLEs			MPSs		
n	Means	Bias	RMSE	Means	Bias	RMSE
30	1.2327	0.0327	0.6077	1.5628	0.3628	0.8418
	6.8249	0.3249	0.8336	6.3810	-0.1190	0.8882
	0.1998	-0.0002	0.0130	0.2081	0.0081	0.0166
100	1.2371	0.0371	0.3557	1.3645	0.1645	0.4200
	6.7095	0.2095	0.5283	6.5274	0.0274	0.4883
	0.2017	0.0017	0.0077	0.2050	0.0050	0.0094
250	1.2247	0.0247	0.2216	1.2816	0.0816	0.2405
	6.6658	0.1658	0.03643	6.5714	0.0714	0.3203
	0.2019	0.0019	0.0051	0.2033	0.0033	0.0057
350	1.2262	0.0262	0.1823	1.2726	0.0726	0.1996
	6.6368	0.1368	0.2905	6.5714	0.0714	0.2586
	0.2018	0.0018	0.0045	0.2030	0.0030	0.0050
1000	1.2259	0.0259	0.1158	1.2486	0.0486	0.1240
	6.6052	0.1052	0.1855	6.5897	0.0897	0.1830
	0.2016	0.0016	0.0033	0.2024	0.0024	0.0038

The results showed that both estimation methods were consistent as the sample size increases from 30 to 1000 since the RMSE decreases and the means converges in probability to the actual values of the parameters. This consistency of MPSE justify the work of [13]. However, for all sample sizes and different actual values of parameters α and γ the MLE proved to be better estimators than MPSE because of their lower RMSEs but for parameter β at sample size from $100 - 1000$, the MPSE was better.

8 Conclusions

This study does not introduce a new distribution, rather an innovation by deriving additional statistical properties of GomR distribution by [8] and estimate the parameters using another method other MLE from the earlier study. The applications were further demonstrated using data sets from another area to ascertained its flexibility over sub models and related distributions. Upon application to rainfall dataset from Nigeria and Malaysia, considering goodness-of-tests statistics, the proposed distribution provides better fit compared to some related distributions although it was the worst in fitting the rainfall data from Argentina. The parameters were estimated using another frequentist method, MPSE. Albeit application to two data sets portray the advantage of MLE over MPSE considering AIC and BIC, simulation study showed that the parameter estimates via both methods were consistent since as the sample size increases, the means converges to the actual values. However those from MLE are more efficient than those from MPSE for majority of the parameters.

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Competing Interests

Authors have declared that no competing interests exist.

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