Asian Journal of Mathematics and Computer Research

Volume 31, Issue 3, Page 20-26, 2024; Article no.AJOMCOR.12185
ISSN: 2395-4205 (P), ISSN: 2395-4213 (O)

Some Unique Fixed Point Theorems on Complete Metric Space with Upper Semi Continuous Mapping

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Authors' contributions

This work was carried out in collaboration among all authors. 'All authors read and approved the final manuscript.

Article Information

DOI: https://doi.org/10.56557/ajomcor/2024/v31i38781

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://prh.ikprress.org/review-history/12185

Received: 15/05/2024
Accepted: 17/07/2024
Published: 18/07/2024

Original Research Article

Abstract

In the present article, we have proved two common fixed point theorems (FPT) for complete metric space (CMS) with the help of upper semi-continuous function along with the existence and uniqueness of fixed point. These results are the generalizations of the well-known theorems Kannan fixed point theorem (KFPT) and Chatterjea fixed point theorem (CFPT) in the existed literature. The KFPT and CFPT are the generalizations of Banach fixed point theorem (BFPT).

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Cite as: *Jojar, Godavari, Uttam Dolhare, Sachin Basude, and Nitin Darkunde. 2024. "Some Unique Fixed Point Theorems on Complete Metric Space With Upper Semi Continuous Mapping". Asian Journal of Mathematics and Computer Research 31 (3):20-26. https://doi.org/10.56557/ajomcor/2024/v31i38781.*

Keywords: CMS; Cauchy sequence; fixed(invariant) point; upper semi continuous mapping.

Mathematics Subject Classification (MSC): 54H25, 47H10.

1 Introduction

Linear and nonlinear are the two primary categories of functional analysis, which is a significant area of mathematics. Because the world in which we live is nonlinear and tiny inputs can have enormous effects, or the other way around, making nonlinear functional analysis a stand-alone.

The criteria that ensure that in a set $\mathcal{X}$ there are points x that can solve the equation $x = \not{f}x$, which is the operator equation, where $\not{f}$ is a transformation specified on a set $\mathcal{X}$, are the subject of fixed point theory. In such problems, the solution set can be of the type either empty, finite, countable, or uncountable infinite. Fixed point theory offers crucial resources for resolving issues that emerge in numerous areas of analysis in mathematics.

2 Preliminaries

BFPT [1] appeared up in 1922, which is supportive in finding answers to the several nonlinear problems that utilized to trim up within the physical, biological, and social sciences, among other scientific spaces.

Definition 2.1 [2]. Consider a set $\mathcal{X}(\neq \phi)$ and the mapping $d: \mathcal{X} \times \mathcal{X} \to [0, \infty)$ which satisfies the below axioms:

(d_1): $d(x_1, x_1) = 0, \forall x_1 \in \mathcal{X}$ (i.e., distance between point to itself is zero)
(d_2): $d(x_1, y_1) = d(y_1, x_1) \Longrightarrow x_1 = y_1, \forall\, x_1, y_1 \in \mathcal{X}$
(d_3): $d(x, y) = d(y, x)$ (which is called symmetric property)
(d_4): $d(x_1, y_1) \leq d(x_1, z_1) + d(z_1, y_1)$, for all $x_1, y_1, z_1 \in \mathcal{X}$ (which is triangle inequality)

Then d is said to be a metric on $\mathcal{X}$ and $(\mathcal{X}, d)$ is called a metric space (MS).

Definition 2.2 [2]. In a MS $(\mathcal{X}, d)$, for a sequence $\{x_n\}$ if $\lim_{n\to\infty} d(x_n, z) = 0 = \lim_{n\to\infty} d(z, x_n)$ then $\{x_n\}$ is said to be convergent to z. In such a case, z is called limit point of the sequence $\{x_n\}$.

Definition 2.3 [2]. Sequence $\{x_n\}$ is a Cauchy sequence in a MS $(\mathcal{X}, d)$, if given $\epsilon > 0$, then there exist $N \in \mathbb{N}$ such that $\forall m, n \geq N$, $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$.

Definition 2.4 [2]. In a MS $(\mathcal{X}, d)$ if every Cauchy sequence $\{x_n\}$ in $\mathcal{X}$ is convergent to a point in $\mathcal{X}$, then $(\mathcal{X}, d)$ is said to be a complete metric space.

If a MS $(\mathcal{X}, d)$ is not complete, then it is called incomplete metric space.

Definition 2.5 [2]. In a MS (X, d) with a self mapping $\not{f}: \mathcal{X} \to \mathcal{X}$ if $\not{f}x = x$, then the point $x \in \mathcal{X}$ is a fixed point of $\not{f}$.

Definition 2.6 [3]. A function $\phi: \mathbb{R} \to [0, \infty)$ is said to be an Upper Semi-Continuous from right if for any sequence $\{x_n\}$ converging to *x* as $x \geq 0$, then $\lim_{n\to\infty} sup\phi(x_n) \leq \phi(x)$.

Theorem 2.1 (Banach) [1]. Let $(\mathcal{X}, d)$ be a CMS and $\not{f}: \mathcal{X} \to \mathcal{X}$ be a self-mapping satisfying the condition $d(\not{f}x, \not{f}y) \leq r d(x, y)$, for all $x, y \in \mathcal{X}$ and $r < 1$ a fixed constant.
Then there exists a unique point $x \in \mathcal{X}$ such that $\not{f}x = x$.

Theorem 2.1 is BFPT, which is also called as *Banach Contraction Principle* (BCP) .

Following the Banach, researchers were left wondering if a map with a fixed point would have a non-contractive type. And Kannan provided the affirmative response in the case of a CMS in 1968 with the below theorem.

Theorem 2.2 (Kannan) [4]. Let $\notin: \mathcal{X} \to \mathcal{X}$, be a self-mapping where $(\mathcal{X}, d)$ is a CMS and $\notin$ satisfies the inequality

$$d(\notin x, \notin y) \leq r[d(x, \notin x) + d(y, \notin y)],$$

where $0 < r < \frac{1}{2}$ and $x, y \in \mathcal{X}$. Then there exists a unique point $x \in \mathcal{X}$ such that $\notin x = x$.

Sarkhel [5,6] used BFPT in 1999 to demonstrate the KFPT. The associated FPT for the Kannan was presented by Chatterjea in 1972 which is as follows:

Theorem 2.3 Chatterjea [7]. Let $(\mathcal{X}, d)$ be a CMS. Let $\notin$ be a Chatterjea mapping on $\mathcal{X}$,

i.e., $\exists\ \alpha \in [0, \frac{1}{2})$ satisfying the following inequality:

$$d(\notin x, \notin y) \leq \alpha(d(x, \notin y) + d(y, \notin x)), \text{ for all } x, y \in \mathcal{X}.$$

Then there is a unique point $x \in \mathcal{X}$ such that $\notin x = x$.

Numerous authors have established the generalizations of the BFPT, KFPT and CFPT in [8,9,10,11] and other places.

The two theorems that will be studied in this paper are the combined generalizations CFPT and the KFPT.

3 Principal Findings

Theorem 3.1. Consider $\notin: \mathcal{X} \to \mathcal{X}$ be a self mapping where $(\mathcal{X}, d)$ is a CMS and $\notin$ satisfies the inequality

$$d(\notin x, \notin y) \leq a_1 \phi\big(d(x, y)\big) + a_2 \phi\left[\frac{d(x, \notin x) d(y, \notin y)}{d(x, y)}\right] \quad \forall x, y \in \mathcal{X} \qquad ...(3.1)$$

for the upper semi continuous from the right function $\phi: \mathbb{R} \to [0, \infty)$ satisfies

$0 \leq \phi(r) < r, \forall\, r > 0$ and $\phi(0) = 0$ with $a_1, a_2 > 0$ and $0 < a_1 + a_2 < 1$.

Then $\exists\,!$ point $z \in \mathcal{X}$ with $\notin z = z$.

Proof. For a arbitrary point $x_0 \in \mathcal{X}$, define the iterative sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ by

$$x_1 = \notin x_0, \qquad x_2 = \notin^2 x_0, \quad x_3 = \notin^3 x_0, \ldots, x_n = \notin x_{n-1} = \notin^n x_0$$

By the condition (3.1) on f, we get

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(\notin x_{n-1}, \notin x_n) \\
&\leq a_1 \phi\big(d(x_{n-1}, x_n)\big) + a_2 \phi\left[\frac{d(x_{n-1}, \notin x_{n-1}) d(x_n, \notin x_n)}{d(x_{n-1}, x_n)}\right] \\
&\leq a_1 \phi\big(d(x_{n-1}, x_n)\big) + a_2 \phi\left[\frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}\right] \\
d(x_n, x_{n+1}) &< a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) \\
[1 - a_2] d(x_n, x_{n+1}) &< a_1 d(x_{n-1}, x_n) \\
d(x_n, x_{n+1}) &< \frac{a_1}{1 - a_2} d(x_{n-1}, x_n) \\
\therefore d(x_n, x_{n+1}) &< R\, d(x_{n-1}, x_n) \qquad \left(\text{where } \; R = \frac{a_1}{1 - a_2}\right)
\end{aligned}$$

Here $0 < R < 1$, because $0 < a_1 + a_2 < 1, \quad a_1, a_2 > 0$.

Proceeding in this way, one will obtain $\quad d(x_n, x_{n+1}) < R^n\, d(x_0, x_1)$

Taking limit as $n \to \infty$, we get $d(x_n, x_{n+1}) \to 0 \quad (\because 0 < R < 1)$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$.

As $(\mathcal{X}, d)$ is a CMS, there exist $z \in \mathcal{X}$ such that $\lim_{n\to\infty} x_n = z$.

We shall prove that $z \in \mathcal{X}$ satisfies the equality $\not{f}z = z$.

Using the continuity of $\not{f}$ we get

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \not{f}x_{n-1} = \not{f}\left(\lim_{n\to\infty} x_{n-1}\right) = \not{f}z$$

Therefore, $\not{f}z = z$ i.e., $\not{f}$ has a fixed point z.

Now, we will prove the uniqueness of z as a fixed point of $\not{f}$.

Let $z_1(\neq z) \in \mathcal{X}$ be other fixed point of $\not{f}$.

Again, by the condition (3.1), we get

$$\begin{aligned} d(z, z_1) &= d(\not{f}z, \not{f}z_1) \\ &\leq a_1\phi\big(d(z, z_1)\big) + a_2\phi\left[\frac{d(z,\not{f}z)d(z_1,\not{f}z_1)}{d(z,z_1)}\right] \\ &\leq a_1\phi\big(d(z, z_1)\big) + a_2\phi\left[\frac{d(z,z)d(z_1,z_1)}{d(z,z_1)}\right] \\ d(z, z_1) &< a_1 d(z, z_1) \end{aligned}$$

Which is obtained of only if $d(z, z_1) = 0 \quad \Rightarrow z = z_1$, which is not true.

Hence, $\exists\,! \, z \in \mathcal{X}$ is such that $\not{f}z = z$.

Example 3.1. Let $\mathcal{X} = \mathbb{R}^+$ and define a function $d: \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$d(x, y) = |x - y| \; \forall x, y \in \mathcal{X}.$$

Then $(\mathcal{X}, d)$ is a CMS. Consider a self-mapping $\not{f}: \mathcal{X} \to \mathcal{X}$, defined by $\not{f}(x) = \frac{x}{2}, \forall x \in \mathcal{X}$ and the function $\phi: \mathbb{R} \to [0, \infty)$ defined by $\phi(r) = \frac{r}{3}$ which is the upper semi continuous from the right with $0 \leq \phi(r) < r, \forall\, r > 0$ and $\phi(0) = 0$, let $a_1 = \frac{1}{2}, a_2 = \frac{1}{4}$, here $a_1 + a_2 < 1$.

Now will check for the inequality (3.1).

LHS = $d(\not{f}x, \not{f}y) = d\left(\frac{x}{6}, \frac{y}{6}\right) = \frac{1}{6}|x - y|$

RHS = $a_1\phi\big(d(x, y)\big) + a_2\phi\left[\frac{d(x,\not{f}x)d(y,\not{f}y)}{d(x,y)}\right]$

$$= \frac{1}{2}\phi(|x - y|) + \frac{1}{4}\phi\left[\frac{\left|x-\frac{x}{6}\right|\left|y-\frac{y}{6}\right|}{|x-y|}\right]$$

$$= \frac{1}{2}\frac{x-y}{3} + \frac{25}{144}\phi\left[\frac{x\cdot y}{x-y}\right]$$

$$= \frac{x-y}{6} + \frac{25}{432}\frac{x\cdot y}{x-y}$$

Here, clearly the inequality (3.1) $d(\not{f}x, \not{f}y) \leq a_1\phi\big(d(x,y)\big) + a_2\phi\left[\frac{d(x,\not{f}x)d(y,\not{f}y)}{d(x,y)}\right] \forall x, y \in \mathcal{X}$ along with all the conditions of theorem 3.1 are satisfied, therefore $\exists\,!$ point $0 \in \mathcal{X}$ with $\not{f}0 = 0$.

Theorem 3.2. On a CMS $(\mathcal{X}, d)$ consider a self-mapping $\not{f}: \mathcal{X} \to \mathcal{X}$ satisfying the following inequality:

$$d(\not{f}x, \not{f}y) \leq a_1\phi\big(d(x,y)\big) + a_2\phi[d(x,\not{f}x) + d(y,\not{f}y)] + a_3\phi[d(x,\not{f}y) + d(y,\not{f}x)] + a_4\phi\left[\frac{d(x,fx)d(y,fy)}{d(x,y)}\right] \quad \text{for all } x, y \in \mathcal{X} \qquad \ldots \qquad (3.2)$$

for the upper semi continuous from the right function $\phi: \mathbb{R} \to [0,\infty)$ with $a_1, a_2, a_3, a_4 > 0$ and $0 < a_1 + 2a_2 + 2a_3 + a_4 < 1$ satisfies

$$0 \leq \phi(r) < r, \forall\, r > 0 \text{ and } \phi(0) = 0$$

Then $\exists\,!$ point $z \in \mathcal{X}$ with $\not{f}(z) = z$.

Proof. For an arbitrary point $x_0 \in \mathcal{X}$, define the iterative sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ by

$$x_1 = \not{f}x_0,\; x_2 = \not{f}^2x_0,\;\; x_3 = \not{f}^3x_0, \ldots, x_n = \not{f}x_{n-1} = \not{f}^nx_0$$

Using the condition (3.2) on $\not{f}$, we obtain

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(\not{f}x_{n-1}, \not{f}x_n) \\
&\leq a_1\phi\big(d(x_{n-1},x_n)\big) + a_2\phi[d(x_{n-1},\not{f}x_{n-1}) + d(x_n,\not{f}x_n)] + \\
&\quad a_3\phi[d(x_{n-1},\not{f}x_n) + d(x_n,\not{f}x_{n-1})] + a_4\phi\left[\frac{d(x_{n-1},\not{f}x_{n-1})d(x_n,\not{f}x_n)}{d(x_{n-1},x_n)}\right] \\
&\leq a_1\phi\big(d(x_{n-1},x_n)\big) + a_2\phi[d(x_{n-1},x_n) + d(x_n,x_{n+1})] \\
&\quad + a_3\phi[d(x_{n-1},x_{n+1}) + d(x_n,x_n)] + a_4\phi\left[\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)}\right] \\
d(x_n,x_{n+1}) &< a_1d(x_{n-1},x_n) + a_2d(x_{n-1},x_n) + a_2d(x_n,x_{n+1}) + \\
&\quad a_3d(x_{n-1},x_n) + a_3d(x_n,x_{n+1}) + a_4d(x_n,x_{n+1}) \\
[1-(a_2+a_3+a_4)]d(x_n,x_{n+1}) &< (a_1+a_2+a_3)d(x_{n-1},x_n) \\
d(x_n,x_{n+1}) &< \frac{a_1+a_2+a_3}{1-(a_2+a_3+a_4)}d(x_{n-1},x_n) \\
\therefore d(x_n,x_{n+1}) &< R\,d(x_{n-1},x_n) \qquad \left(\text{where } R = \frac{a_1+a_2+a_3}{1-(a_2+a_3+a_4)}\right)
\end{aligned}$$

Here $0 < R < 1$, because $0 < a_1 + 2a_2 + 2a_3 + a_4 < 1$, $a_1, a_2, a_3, a_4 > 0$.

Proceeding in this manner, we obtain $d(x_n, x_{n+1}) < R^n\, d(x_0, x_1)$.

Taking limit as $n \to \infty$, we get $d(x_n, x_{n+1}) \to 0$ $\quad(\because 0 < R < 1)$

Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy in $\mathcal{X}$.

As the metric space $(\mathcal{X}, d)$ is complete, so there exist $z \in \mathcal{X}$ such that $\lim_{n\to\infty} x_n = z$.

We will prove that z is a fixed point of $\not{f}$.
As the function $\not{f}$ is continuous, so we get

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \not{f}x_{n-1} = \not{f}\left(\lim_{n\to\infty} x_{n-1}\right) = \not{f}z$$

Therefore, $\not{f}z = z$ and hence z is a fixed point of $\not{f}$.

Now, we will prove the uniqueness of z as a fixed point of $\not{f}$.

Let $z_1(\neq z) \in \mathcal{X}$ be other fixed point of $\notin$.

Again, by the condition (3.2), we get

$$\begin{aligned} d(z, z_1) &= d(\notin z, \notin z_1) \\ &\leq a_1\phi\big(d(z, z_1)\big) + a_2\phi[d(z, \notin z) + d(z_1, \notin z_1)] + a_3\phi[d(z, \notin z_1) + \\ &\quad d(z_1, \notin z)] + a_4\phi\left[\frac{d(z,\notin z)d(z_1,\notin z_1)}{d(z,z_1)}\right] \\ &\leq a_1\phi\big(d(z, z_1)\big) + a_2\phi[d(z, z) + d(z_1, z_1)] + a_3\phi[d(z, z_1) + \ d(z_1, z)] + \\ &\quad a_4\phi\left[\frac{d(z,z)d(z_1,z_1)}{d(z,z_1)}\right] \\ &\leq a_1\phi\big(d(z, z_1)\big) + a_3\phi[d(z, z_1) + d(z_1, z)] \\ &\leq a_1\phi\big(d(z, z_1)\big) + a_3\phi[2d(z, z_1)] \\ d(z, z_1) &< (a_1 + 2a_3)d(z, z_1) \end{aligned}$$

Which is obtained of only if $d(z, z_1) = 0 \quad \Rightarrow z = z_1$,

which is a not possible by our assumption.

Hence, $\exists\,!\, z \in \mathcal{X}$ such that $\notin z = z$.

Example 3.2. Let $\mathcal{X} = \mathbb{R}^+$ and define a function $d: \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$d(x, y) = |x - y|\ \forall x, y \in \mathcal{X}$.

Then $(\mathcal{X}, d)$ is a CMS. Consider a self-mapping $\notin: \mathcal{X} \to \mathcal{X}$, defined by $\notin(x) = \frac{x}{5}, \forall x \in \mathcal{X}$ and the function $\phi: \mathbb{R} \to [0, \infty)$ defined by $\phi(r) = \frac{r}{2}$ which is the upper semi continuous from the right with $0 \leq \phi(r) < r, \forall\, r > 0$ and $\phi(0) = 0$, let $a_1 = a_2 = a_3 = a_4 = \frac{1}{6}, a_2 = \frac{1}{4}$,

here $a_1 + a_2 + a_3 + a_4 < 1$.

Now will check for the inequality (3.2).

$$\text{LHS} = d(\notin x, \notin y) = d\left(\frac{x}{5}, \frac{y}{5}\right) = \frac{1}{5}|x - y|$$

$$\begin{aligned} \text{RHS} &= a_1\phi\big(d(x, y)\big) + a_2\phi[d(x, \notin x) + d(y, \notin y)] + a_3\phi[d(x, \notin y) + d(y, \notin x)] + a_4\phi\left[\frac{d(x,fx)d(y,fy)}{d(x,y)}\right] \\ &= \frac{1}{12}|x - y| + \frac{1}{12}\left[\left|x - \frac{x}{5}\right| + \left|y - \frac{y}{5}\right|\right] + + \frac{1}{12}\left[\left|x - \frac{y}{5}\right| + \left|y - \frac{x}{5}\right|\right] + \frac{1}{12}\left[\frac{\left|x-\frac{x}{5}\right|\cdot\left|y-\frac{y}{5}\right|}{|x-y|}\right] \\ &= \frac{1}{12}|x - y| + \frac{1}{12}\left[\frac{4x}{5} + \frac{4y}{5}\right] + \frac{1}{12}\left[\frac{4y}{5} + \frac{4x}{5}\right] + \frac{1}{12}\left[\frac{\frac{4x}{5}\cdot\frac{4y}{5}}{|x-y|}\right] \\ &= \frac{1}{12}|x - y| + \frac{1}{15}[x + y] + \frac{1}{15}[y + x] + \frac{1}{12}\left[\frac{x\cdot y}{|x-y|}\right] \end{aligned}$$

Here, clearly the inequality (3.2) $d(\notin x, \notin y) \leq a_1\phi\big(d(x, y)\big) + a_2\phi[d(x, \notin x) + d(y, \notin y)] + a_3\phi[d(x, \notin y) + d(y, \notin x)] + a_4\phi\left[\frac{d(x,fx)d(y,fy)}{d(x,y)}\right]$ $\forall x, y \in \mathcal{X}$ along with all the conditions of theorem 3.1 are satisfied, therefore $\exists\,!$ point $0 \in \mathcal{X}$ with $\notin 0 = 0$.

4 Conclusion

In this study, we used upper semi continuous mapping to prove the uniqueness and existence of common fixed point theorems for CMS. The BFPT, KFPT and CFPT found in previous research are all generalized by our findings. Also, we have given examples which will give the justice to our main findings.

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Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Acknowledgement

First author acknowledges Chhatrapati Shahu Maharaj Research Training and Human Development Institute (SARTHI), Pune for providing the CSMNRF-20 fellowship for carrying out this research work.

Competing Interests

Authors have declared that no competing interests exist.

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