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Applications of Liouville's Identity with an Odd Function

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Abstract

In this paper, we are based on the Huard, Ou, Spearman and Williams's generalization of Liouville's Identity so we obtain

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a<b}} a = \frac{1}{2}\left(\sigma_2(n)+\sigma_1(n)-2n\sigma_0(n)\right), \qquad \sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x,y\text{odd}}} a = \frac{n}{2}\sigma_0(\frac{n}{2}),$$

and etc. Also, independently we attempt to consider the Liouville's Identity, therefore as the application of his identity, we have the restricted combinatoric convolution sums as

$$\sum_{i=0}^{l}\binom{2l+1}{2i+1}\sum_{\substack{(a,b,x)\in\mathbb{N}^3\\ (a+b)x=n\\ x\equiv m(\bmod 2m)}} a^{2l-2i}b^{2i+1}$$
$$= \frac{1}{2}\left\{\sigma_{2l+2}(\frac{n}{m})-\sigma_{2l+1}(\frac{n}{m})-\sigma_{2l+2}(\frac{n}{2m})+\sigma_{2l+1}(\frac{n}{2m})\right\},$$
$$\sum_{i=0}^{l}\binom{2l+1}{2i+1}\sum_{\substack{(a,b,x)\in\mathbb{N}^3\\ (a+b)x=n\\ a\equiv m(\bmod 2m)\\ b\equiv m(\bmod 2m)}} a^{2l-2i}b^{2i+1} = \frac{(2m)^{2l+1}}{2}\sigma_{2l+2}(\frac{n}{2m}),$$

(see Theorem 1.6 and Theorem 1.8) and etc., by dealing with an odd function for $m, n \in \mathbb{N}$ and $l \in \mathbb{N}\cup\{0\}$.

Keywords: Divisor functions, Convolution sums

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1 Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ will denote the sets of positive integers, rational integers, and complex numbers, respectively. Let r be a nonnegative integer and d, n be positive integers. The sum of divisors function $\sigma_r(n)$ is defined by

$$\sigma_r(n) = \begin{cases} \sum_{d|n} d^r, & \text{if } n \in \mathbb{N}\ , \\ 0, & \text{if } n \in \mathbb{Q}, \quad n \notin \mathbb{N}. \end{cases}$$

The Bernoulli numbers B_n $(n \in \mathbb{N}_0)$ are defined by the recurrence relation

$$B_n = -\frac{1}{n+1}\sum_{j=0}^{n-1}\binom{n+1}{j}B_j, \quad n \in \mathbb{N},$$

where $B_0 = 1$, or equivalently by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad x \in \mathbb{R}, \quad |x| < 2\pi.$$

The first few Bernoulli numbers are $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$, $B_7 = 0$, $B_8 = -1/30$, $B_9 = 0$, $B_{10} = 5/66$, $B_{11} = 0$, $B_{12} = -691/2730$. Indeed $B_{2n+1} = 0$, $n \in \mathbb{N}$, see for example Berndt's book ([1], Theorem 4.1.2, p. 85).

The history of the convolution sums involving the divisor functions $\sigma_r(n)$ goes back to Glaisher [2], [3], [4]. Many recent works on convolution formulas for divisor functions can be found in B. C. Berndt [5]; H. Hahn [6]; J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams [7]; G. Melfi [8]; B. Cho, D. Kim, and J.-K. Koo [9], [10]; and A. Alaca, S. Alaca, and K. S. Williams [11], [12].

In Section 2, after calculating

$$\sum_{\substack{(a,x,y)\in\mathbb{N}^3 \\ a(x+y)=n}} a \quad \text{and} \quad \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a<b}} a,$$

we obtain

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3 \\ (a+b)x=n}} a$$

since

$$\begin{aligned} &\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a<b}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a=b}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a>b}} a \\ = &\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ x>y}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ x=y}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ x<y}} a, \end{aligned}$$

that is,

Theorem 1.1. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n}} a = n\sigma_0(n) - \sigma_1(n).$$

(b)

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a<b}} a = \frac{1}{2}\left(\sigma_2(n) + \sigma_1(n) - 2n\sigma_0(n)\right).$$

And in Section 3, we are motivated by Liouville's Identity so we try to obtain some convolution sums formulae, especially, related to an odd function with the various conditions as follows :

Theorem 1.2. *Let $n \in \mathbb{N}$ with $l \in \mathbb{N} \cup \{0\}$. Then we have*

$$\begin{aligned}
&\sum_{i=0}^{l} \binom{2l+1}{2i+1} \sum_{j=0}^{2i+1} (-1)^j \binom{2i+1}{j} n^{2i+1-j} \\
&\quad \times \frac{1}{2l-2i+j+1} \sum_{m=0}^{2l-2i+j} (-1)^m \binom{2l-2i+j+1}{m} B_m (n-1)^{2l-2i+j+1-m} \\
&= \frac{1}{2}\left(n^{2l+2} - n^{2l+1}\right)
\end{aligned}$$

with Bernoulli number B_m.

Theorem 1.3. *Let $n \in \mathbb{N}$ with $l \in \mathbb{N} \cup \{0\}$. Then*

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3\\ (a+b)x=n}} \cos a \sin b = \frac{1}{2} \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} \left(\sigma_{2l+2}(n) - \sigma_{2l+1}(n)\right).$$

Theorem 1.4. *Let $m, n \in \mathbb{N}$ with $l \in \mathbb{N} \cup \{0\}$. Then*

$$\sum_{i=0}^{l} \binom{2l+1}{2i+1} \sum_{\substack{(a,b,x)\in\mathbb{N}^3\\ (a+b)x=n\\ x\equiv 0 (\bmod m)}} a^{2l-2i} b^{2i+1} = \frac{1}{2}\left\{\sigma_{2l+2}(\frac{n}{m}) - \sigma_{2l+1}(\frac{n}{m})\right\}.$$

Theorem 1.5. *Let $m, n \in \mathbb{N}$ with $l \in \mathbb{N} \cup \{0\}$. Then*

$$\begin{aligned}
&\sum_{i=0}^{l} \binom{2l+1}{2i+1} \sum_{\substack{(a,b,x)\in\mathbb{N}^3\\ (a+b)x=n\\ x\not\equiv 0 (\bmod m)}} a^{2l-2i} b^{2i+1} \\
&= \frac{1}{2}\left\{\sigma_{2l+2}(n) - \sigma_{2l+1}(n) - \sigma_{2l+2}(\frac{n}{m}) + \sigma_{2l+1}(\frac{n}{m})\right\}.
\end{aligned}$$

Theorem 1.6. *Let* $m, n \in \mathbb{N}$ *with* $l \in \mathbb{N} \cup \{0\}$. *Then*

$$\sum_{i=0}^{l} \binom{2l+1}{2i+1} \sum_{\substack{(a,b,x)\in\mathbb{N}^3 \\ (a+b)x=n \\ x\equiv m(\bmod 2m)}} a^{2l-2i} b^{2i+1}$$
$$= \frac{1}{2}\left\{\sigma_{2l+2}(\frac{n}{m}) - \sigma_{2l+1}(\frac{n}{m}) - \sigma_{2l+2}(\frac{n}{2m}) + \sigma_{2l+1}(\frac{n}{2m})\right\}.$$

Theorem 1.7. *Let* $m, n \in \mathbb{N}$ *with* $l \in \mathbb{N} \cup \{0\}$. *Then*

$$\sum_{i=0}^{l} \binom{2l+1}{2i+1} \sum_{\substack{(a,b,x)\in\mathbb{N}^3 \\ (a+b)x=n \\ a\equiv 0(\bmod m) \\ b\equiv 0(\bmod m)}} a^{2l-2i} b^{2i+1} = \frac{m^{2l+1}}{2}\left\{\sigma_{2l+2}(\frac{n}{m}) - \sigma_{2l+1}(\frac{n}{m})\right\}.$$

Theorem 1.8. *Let* $m, n \in \mathbb{N}$ *with* $l \in \mathbb{N} \cup \{0\}$. *Then*

$$\sum_{i=0}^{l} \binom{2l+1}{2i+1} \sum_{\substack{(a,b,x)\in\mathbb{N}^3 \\ (a+b)x=n \\ a\equiv m(\bmod 2m) \\ b\equiv m(\bmod 2m)}} a^{2l-2i} b^{2i+1} = \frac{(2m)^{2l+1}}{2}\sigma_{2l+2}(\frac{n}{2m}).$$

2 Application to an Identity of Huard, Ou, Spearman and Williams

In 2000 Huard, Ou, Spearman and Williams proved a far reaching generalization of Liouville's Identity.

Proposition 2.1. *([13], Theorem 13.1) Let* $f : \mathbb{Z}^4 \to \mathbb{C}$ *be such that*

$$f(a,b,x,y) - f(x,y,a,b) = f(-a,-b,x,y) - f(x,y,-a,-b) \tag{2.1}$$

for all integers a, b, x *and* y. *Let* $n \in \mathbb{N}$. *Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (f(a,b,x,-y) - f(a,-b,x,y) + f(a,a-b,x+y,y)$$
$$- f(a,a+b,y-x,y) + f(b-a,b,x,x+y) - f(a+b,b,x,x-y))$$
$$= \sum_{\substack{d\in\mathbb{N} \\ d|n}} \sum_{\substack{x\in\mathbb{N} \\ x<d}} (f(0,\frac{n}{d},x,d) + f(\frac{n}{d},0,d,x) + f(\frac{n}{d},\frac{n}{d},d-x,-x)$$
$$- f(x,x-d,\frac{n}{d},\frac{n}{d}) - f(x,d,0,\frac{n}{d}) - f(d,x,\frac{n}{d},0)).$$

Proof of Theorem 1.1. (a) We note that

$$\sum_{\substack{(a,x,y)\in\mathbb{N}^3 \\ a(x+y)=n}} a = \sum_{a|n} \frac{n}{a}(a-1) = \sum_{a|n}\left(n - \frac{n}{a}\right)$$
$$= n\sum_{a|n} a^0 - \sum_{e|n} e = n\sigma_0(n) - \sigma_1(n)$$

where we put $e := \frac{n}{a}$.

(b) Let us take $f(a,b,x,y) = |a|$ in Proposition 2.1. Then (2.1) is satisfied, so the left hand side becomes

$$\begin{aligned}
&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (|b-a|-|a+b|)\\
&= \sum_{\substack{ax+by=n\\ b\geq a}} \{(b-a)-(a+b)\} + \sum_{\substack{ax+by=n\\ b<a}} \{-(b-a)-(a+b)\}\\
&= -2\sum_{\substack{ax+by=n\\ b>a}} a - 2\sum_{\substack{ax+by=n\\ b=a}} a - 2\sum_{\substack{ax+by=n\\ b<a}} b\\
&= -4\sum_{\substack{ax+by=n\\ b>a}} a - 2\sum_{\substack{ax+by=n\\ b=a}} a.
\end{aligned} \tag{2.2}$$

Then using Theorem 1.1 (a) for the second term of (2.2), we can write (2.2) as

$$-4\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ b>a}} a - 2\{n\sigma_0(n)-\sigma_1(n)\}. \tag{2.3}$$

However, the right hand side of Proposition 2.1 is

$$\sum_{\substack{d\in\mathbb{N}\\ d|n}}\sum_{\substack{x\in\mathbb{N}\\ x<d}} \left\{\left|\frac{n}{d}\right| + \left|\frac{n}{d}\right| - |x| - |x| - |d|\right\} = \sum_{\substack{d\in\mathbb{N}\\ d|n}}\sum_{\substack{x\in\mathbb{N}\\ x<d}} \left\{2\frac{n}{d} - 2x - d\right\}. \tag{2.4}$$

Then the first term in (2.4) is

$$\begin{aligned}
2\sum_{\substack{d\in\mathbb{N}\\ d|n}}\sum_{\substack{x\in\mathbb{N}\\ x<d}} \frac{n}{d} &= 2\sum_{d|n}\frac{n}{d}\sum_{x=1}^{d-1} 1 = 2\sum_{d|n}\frac{n}{d}(d-1) = 2\sum_{d|n}\left(n-\frac{n}{d}\right)\\
&= 2n\sum_{d|n} 1 - 2\sum_{d|n}\frac{n}{d} = 2n\sigma_0(n) - 2\sum_{e|n} e = 2n\sigma_0(n) - 2\sigma_1(n),
\end{aligned} \tag{2.5}$$

where we put $e := \frac{n}{d}$. And the second term in (2.4) is

$$\begin{aligned}
-2\sum_{\substack{d\in\mathbb{N}\\ d|n}}\sum_{\substack{x\in\mathbb{N}\\ x<d}} x &= -2\sum_{d|n}\sum_{x=1}^{d-1} x = -2\sum_{d|n}\frac{1}{2}(d-1)d = -\sum_{d|n}\left(d^2-d\right)\\
&= -\sigma_2(n) + \sigma_1(n).
\end{aligned}$$

Finally, the third term in (2.4) is

$$-\sum_{\substack{d\in\mathbb{N}\\ d|n}}\sum_{\substack{x\in\mathbb{N}\\ x<d}} d = -\sum_{d|n} d\sum_{x=1}^{d-1} 1 = -\sum_{d|n} d(d-1) = -\sigma_2(n) + \sigma_1(n).$$

Therefore (2.4) becomes

$$2n\sigma_0(n) - 2\sigma_2(n). \tag{2.6}$$

Equating (2.3) with (2.6) we obtain the formula

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ b>a}} a = \frac{1}{2}\left(\sigma_2(n) + \sigma_1(n) - 2n\sigma_0(n)\right).$$

□

To deduce Corollary 2.3 we consider as follows

$$\begin{aligned}&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a<b}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a=b}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a>b}} a\\ &= \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x>y}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x=y}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x<y}} a.\end{aligned} \tag{2.7}$$

Corollary 2.1. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x>y}} a = \frac{1}{2}\left(\sigma_2(n) + \sigma_1(n) - 2n\sigma_0(n)\right).$$

Proof. Let us change the summation of Theorem 1.1 (b) as

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a<b}} a = \sum_{ax+(a+b)y=n} a = \sum_{a(x+y)+by=n} a = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x>y}} a. \tag{2.8}$$

So

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x>y}} a = \frac{1}{2}\left(\sigma_2(n) + \sigma_1(n) - 2n\sigma_0(n)\right).$$

□

Lemma 2.2. *We obtain*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a>b}} a = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x<y}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x>y}} a.$$

Proof. We note that

$$\begin{aligned}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a>b}} a &= \sum_{(a+b)x+by=n} (a+b) = \sum_{(a+b)x+by=n} a + \sum_{(a+b)x+by=n} b\\ &= \sum_{ax+b(x+y)=n} a + \sum_{\substack{ax+by=n\\ a>b}} b = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x<y}} a + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a<b}} a.\end{aligned} \tag{2.9}$$

Here the second term in (2.9) is equal to $\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x>y}} a$ by (2.8). So the proof is complete. □

Corollary 2.3. *Let* $n \in \mathbb{N}$. *Then we have*

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3\\ (a+b)x=n}} a = \frac{1}{2}\left(\sigma_2(n) - \sigma_1(n)\right).$$

Proof. By (2.7), (2.8) and Lemma 2.2, we obtain

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ a=b}} a = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x=y}} a - \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x>y}} a.$$

So we use Theorem 1.1 (a) and Corollary 2.1.

□

Theorem 2.4. *Let* $n \in \mathbb{N}$. *Then we have*

(a)

$$\sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x,y \text{ odd}}} a = \frac{n}{2}\sigma_0(\frac{n}{2}).$$

(b)

$$\sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x,y \text{ even}}} a = \frac{n}{2}\sigma_0(\frac{n}{2}) - \sigma_1(\frac{n}{2}).$$

(c)

$$\sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x \text{ odd}, y \text{ even}}} a = \frac{1}{2}\left\{n\sigma_0(n) - n\sigma_0(\frac{n}{2}) - \sigma_1(n) + \sigma_1(\frac{n}{2})\right\}.$$

Proof. (a) We observe that

$$\sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x,y \text{ odd}}} a = \sum_{a(2x+2y-2)=n} a = \sum_{a(x+y-1)=\frac{n}{2}} a = \sum_{a|\frac{n}{2}} \frac{n/2}{a}\cdot a$$
$$= \frac{n}{2}\sum_{a|\frac{n}{2}} a^0 = \frac{n}{2}\sigma_0(\frac{n}{2}).$$

(b) By using Theorem 1.1 (a), we obtain that

$$\sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x,y \text{ even}}} a = \sum_{a(2x+2y)=n} a = \sum_{a(x+y)=\frac{n}{2}} a = \frac{n}{2}\sigma_0(\frac{n}{2}) - \sigma_1(\frac{n}{2}).$$

(c) Since

$$\begin{aligned}\sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n}} a &= \sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x,y \text{ odd}}} a + \sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x,y \text{ even}}} a + \sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x \text{ odd}, y \text{ even}}} a + \sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x \text{ even}, y \text{ odd}}} a\\ &= \sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x,y \text{ odd}}} a + \sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x,y \text{ even}}} a + 2\sum_{\substack{(a,x,y)\in\mathbb{N}^3\\ a(x+y)=n\\ x \text{ odd}, y \text{ even}}} a,\end{aligned}$$

we refer to Theorem 1.1 (a) and Theorem 2.4 (a), (b).

□

3 Application of Liouville identity

Using the four basic theta functions, one is lead to the formula (see ([13], p. 14))(for $k, n \in \mathbb{N}$),

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=2n\\ a,b,x,y \text{ odd}}} \left((a+b)^{2k} - (a-b)^{2k}\right) = \sum_{\substack{m\in\mathbb{N}\\ m|n\\ \frac{n}{m} \text{ odd}}} 2^{2k} m^{2k+1}. \tag{3.1}$$

There are several arithmetical formulae due to Liouville. A classical one of Liouville very often used in elementary number theory (completely avoiding the hard analysis) is that (see ([14], p. 144)) : For $n \in \mathbb{N}$ and for $f : \mathbb{Z} \to \mathbb{C}$ an even function, the identity

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=2n\\ a,b,x,y \text{ odd}}} (f(a+b) - f(a-b)) = \sum_{\substack{m\in\mathbb{N}\\ m|n\\ \frac{n}{m} \text{ odd}}} (f(2m) - f(0)) \tag{3.2}$$

holds.

An another important classical formula of Liouville is ([15], p. 247) : For $n \in \mathbb{N}$ and for $f : \mathbb{Z} \to \mathbb{C}$ an even function, the identity

$$\begin{aligned}&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (f(a-b) - f(a+b))\\ &= f(0)\left(\sigma_1(n) - \sigma_0(n)\right) + \sum_{\substack{d\in\mathbb{N}\\ d|n}} \left(1 + \frac{2n}{d} - d\right) f(d) - 2\sum_{\substack{d\in\mathbb{N}\\ d|n}} \left(\sum_{v=1}^{d} f(v)\right).\end{aligned} \tag{3.3}$$

Lemma 3.1. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b)\in\mathbb{N}^2\\ a+b=n}} (F(a-b) - F(a+b)) = -(n-1)F(n). \tag{3.4}$$

Proof. Proof begins by collecting together terms on the left hand side having a common value $F(k)$. When $k = 0$ there is a single term, namely,

$$F(0) \sum_{\substack{a+b=n\\ a=b}} 1,$$

and when $k \geq 1$ we have

$$F(k)\left(\sum_{\substack{a+b=n\\a-b=k}} 1 - \sum_{\substack{a+b=n\\a-b=-k}} 1\right) - \sum_{a+b=n} F(n) = -F(n)\sum_{a+b=n} 1$$

as F is an odd function.
Therefore we can write the left-hand side of the identity as

$$\begin{aligned}\sum_{a+b=n}(F(a-b)-F(a+b)) &= F(0)\sum_{\substack{a+b=n\\a=b}} 1 - F(n)\sum_{a+b=n} 1\\ &= F(0)\sum_{2a=n} 1 - F(n)(n-1)\\ &= F(0)\sum_{a=\frac{n}{2}} 1 - (n-1)F(n)\\ &= F(0)-(n-1)F(n).\end{aligned} \tag{3.5}$$

However we recall that a function $F : \mathbb{Z} \to \mathbb{C}$ is called an odd function if

$$F(-x) = -F(x)$$

for all integers x. This shows that $F(0) = 0$. Thus Eq. (3.5) becomes

$$\sum_{a+b=n}(F(a-b)-F(a+b)) = -(n-1)F(n).$$

So the proof is complete.

□

Proof of Theorem 1.2. We apply $f(x) = x^{2l+1}$ in Lemma 3.1. Then the left hand side is

$$\begin{aligned}&\sum_{a+b=n}(F(a-b)-F(a+b)) = \sum_{a+b=n}\left((a-b)^{2l+1}-(a+b)^{2l+1}\right)\\ &= \sum_{a+b=n}\left(\sum_{i=0}^{2l+1}\binom{2l+1}{i}(-1)^i a^{2l+1-i}b^i - \sum_{i=0}^{2l+1}\binom{2l+1}{i}a^{2l+1-i}b^i\right)\\ &= -2\sum_{a+b=n}\sum_{\substack{i=0\\ i \text{ odd}}}^{2l+1}\binom{2l+1}{i}a^{2l+1-i}b^i\\ &= -2\sum_{i=0}^{l}\binom{2l+1}{2i+1}\sum_{a+b=n}a^{2l-2i}b^{2i+1}.\end{aligned} \tag{3.6}$$

Here we can observe that

$$\sum_{a+b=n} a^{2l-2i}b^{2i+1} = \sum_{a=1}^{n-1} a^{2l-2i}(n-a)^{2i+1} = \sum_{a=1}^{n-1} a^{2l-2i} \sum_{j=0}^{2i+1} \binom{2i+1}{j} n^{2i+1-j}(-1)^j a^j$$
$$= \sum_{j=0}^{2i+1} \binom{2i+1}{j} (-1)^j n^{2i+1-j} \sum_{a=1}^{n-1} a^{2l-2i+j}$$
$$= \sum_{j=0}^{2i+1} \binom{2i+1}{j} (-1)^j n^{2i+1-j} \frac{1}{2l-2i+j+1} \sum_{m=0}^{2l-2i+j} (-1)^m \binom{2l-2i+j+1}{m}$$
$$\times B_m(n-1)^{2l-2i+j+1-m},$$

where we use the Faulhaber's formula as

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j \binom{p+1}{j} B_j n^{p+1-j}$$

with the Bernoulli number B_j. So Eq. (3.6) becomes

$$\sum_{a+b=n} (F(a-b) - F(a+b))$$
$$= -2 \sum_{i=0}^{l} \binom{2l+1}{2i+1} \sum_{j=0}^{2i+1} \binom{2i+1}{j} (-1)^j n^{2i+1-j} \quad (3.7)$$
$$\times \frac{1}{2l-2i+j+1} \sum_{m=0}^{2l-2i+j} (-1)^m \binom{2l-2i+j+1}{m} B_m (n-1)^{2l-2i+j+1-m}.$$

And the right hand side of (3.4) is

$$-(n-1)F(n) = -(n-1)n^{2l+1} = -n^{2l+2} + n^{2l+1}. \quad (3.8)$$

Equating (3.7) and (3.8) we obtain the proof.

□

Corollary 3.2. *We have*

$$\sum_{i=0}^{l} \binom{2l+1}{2i+1} \sum_{j=0}^{2i+1} (-1)^j \binom{2i+1}{j} 2^{2i+1-j}$$
$$\times \frac{1}{2l-2i+j+1} \sum_{m=0}^{2l-2i+j} (-1)^m \binom{2l-2i+j+1}{m} B_m$$
$$= 2^{2l}.$$

Proof. It is an immediate consequence of Theorem 1.2 with $n = 2$.

□

Lemma 3.3. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3 \\ (a+b)x=n}} (F(a-b) - F(a+b)) = -\sum_{k|n} (k-1)F(k).$$

Proof. We prove by solving terms on the left hand side having a common value $F(k)$. When $k = 0$ then,

$$F(0) \sum_{\substack{(a+b)x=n \\ a=b}} 1,$$

and when $k \geq 1$ we obtain

$$F(k)\left(\sum_{\substack{(a+b)x=n \\ a-b=k}} 1 - \sum_{\substack{(a+b)x=n \\ a-b=-k}} 1 - \sum_{\substack{(a+b)x=n \\ a+b=k}} 1\right) = -F(k) \sum_{\substack{(a+b)x=n \\ a+b=k}} 1,$$

as F is an odd function.
Therefore the left-hand side of the identity can be written as

$$\begin{aligned}
\sum_{(a+b)x=n} (F(a-b) - F(a+b)) &= F(0) \sum_{\substack{(a+b)x=n \\ a=b}} - \sum_{k\geq 1} F(k) \sum_{\substack{(a+b)x=n \\ a+b=k}} 1 \\
&= F(0) \sum_{2ax=n} 1 - \sum_{\substack{k\geq 1 \\ k|n}} F(k)\,(k-1) \\
&= F(0) \sum_{a|\frac{n}{2}} 1 - \sum_{k|n} (k-1)\,F(k) \\
&= F(0)\sigma_0(\frac{n}{2}) - \sum_{k|n} (k-1)F(k).
\end{aligned}$$

Since F is an odd function, we have $F(0) = 0$. Therefore

$$\sum_{(a+b)x=n} (F(a-b) - F(a+b)) = -\sum_{k|n} (k-1)F(k).$$

□

Proof of Theorem 1.3. Let $F(x) = \sin x$ in Lemma 3.3. Then the left hand side is

$$\begin{aligned}
\sum_{(a+b)x=n} (F(a-b) - F(a+b)) &= \sum_{(a+b)x=n} (\sin(a-b) - \sin(a+b)) \\
&= -2 \sum_{(a+b)x=n} \cos a \sin b.
\end{aligned} \tag{3.9}$$

And the right hand side is

$$\begin{aligned}
-\sum_{k|n} (k-1)\sin(k) &= -\sum_{k|n} (k-1) \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} k^{2l+1} \\
&= -\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} \sum_{k|n} (k-1) k^{2l+1} \\
&= -\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} \left(\sigma_{2l+2}(n) - \sigma_{2l+1}(n)\right),
\end{aligned} \tag{3.10}$$

where we use the Taylor series $\sin x = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} x^{2l+1}$.
Equating (3.9) and (3.10) we obtain the proof.

□

Now we specify Lemma 3.3 according to the condition of x, a, and b.

Lemma 3.4. *Let $m, n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3 \\ (a+b)x=n \\ x\equiv 0(\bmod m)}} (F(a-b) - F(a+b)) = -\sum_{k|\frac{n}{m}} (k-1)F(k).$$

Proof. We note that

$$\begin{aligned}\sum_{\substack{(a+b)x=n \\ x\equiv 0(\bmod m)}} (F(a-b) - F(a+b)) &= \sum_{(ma+mb)x=n} (F(a-b) - F(a+b)) \\ &= \sum_{(a+b)x=\frac{n}{m}} (F(a-b) - F(a+b)) \\ &= -\sum_{k|\frac{n}{m}} (k-1)F(k),\end{aligned}$$

since

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3 \\ (a+b)x=n}} (F(a-b) - F(a+b)) = -\sum_{k|n} (k-1)F(k)$$

in Lemma 3.3. □

Proof of Theorem 1.4. Let $f(x) = x^{2l+1}$ in Lemma 3.4. Then the left hand side of Lemma 3.4 is

$$\begin{aligned}&\sum_{\substack{(a+b)x=n \\ x\equiv 0(\bmod m)}} (F(a-b) - F(a+b)) \\ &= \sum_{\substack{(a+b)x=n \\ x\equiv 0(\bmod m)}} \left((a-b)^{2l+1} - (a+b)^{2l+1}\right) \\ &= \sum_{\substack{(a+b)x=n \\ x\equiv 0(\bmod m)}} \left(\sum_{i=0}^{2l+1} \binom{2l+1}{i} a^{2l+1-i}(-b)^i - \sum_{i=0}^{2l+1} \binom{2l+1}{i} a^{2l+1-i} b^i\right) \\ &= -2 \sum_{\substack{(a+b)x=n \\ x\equiv 0(\bmod m)}} \sum_{i=0}^{l} \binom{2l+1}{2i+1} a^{2l-2i} b^{2i+1} \\ &= -2 \sum_{i=0}^{l} \binom{2l+1}{2i+1} \sum_{\substack{(a+b)x=n \\ x\equiv 0(\bmod m)}} a^{2l-2i} b^{2i+1}.\end{aligned} \tag{3.11}$$

And the right hand side of Lemma 3.4 is

$$-\sum_{k|\frac{n}{m}}(k-1)F(k) = -\sum_{k|\frac{n}{m}}(k-1)k^{2l+1} = -\left\{\sigma_{2l+2}(\frac{n}{m}) - \sigma_{2l+1}(\frac{n}{m})\right\}. \tag{3.12}$$

Equating (3.11) with (3.12) we obtain

$$\sum_{i=0}^{l}\binom{2l+1}{2i+1}\sum_{\substack{(a,b,x)\in\mathbb{N}^3\\(a+b)x=n\\x\equiv 0(\bmod m)}} a^{2l-2i}b^{2i+1} = \frac{1}{2}\left\{\sigma_{2l+2}(\frac{n}{m}) - \sigma_{2l+1}(\frac{n}{m})\right\}.$$

□

Lemma 3.5. *Let $m, n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3\\(a+b)x=n\\x\not\equiv 0(\bmod m)}} (F(a-b) - F(a+b)) = -\sum_{k|n}(k-1)F(k) + \sum_{k|\frac{n}{m}}(k-1)F(k).$$

Proof. We can write

$$\begin{aligned}&\sum_{\substack{(a+b)x=n\\x\not\equiv 0(\bmod m)}} (F(a-b) - F(a+b))\\ &= \sum_{(a+b)x=n} (F(a-b) - F(a+b)) - \sum_{\substack{(a+b)x=n\\x\equiv 0(\bmod m)}} (F(a-b) - F(a+b))\\ &= -\sum_{k|n}(k-1)F(k) + \sum_{k|\frac{n}{m}}(k-1)F(k),\end{aligned}$$

by Lemma 3.3 and Lemma 3.4.

□

Proof of Theorem 1.5. Proof is similar to Theorem 1.4 except for letting $f(x) = x^{2l+1}$ in Lemma 3.5. □

Lemma 3.6. *Let $m, n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3\\(a+b)x=n\\x\equiv m(\bmod 2m)}} (F(a-b) - F(a+b)) = -\sum_{k|\frac{n}{m}}(k-1)F(k) + \sum_{k|\frac{n}{2m}}(k-1)F(k).$$

Proof. Using Lemma 3.4, we obtain

$$\begin{aligned}&\sum_{\substack{(a+b)x=n\\x\equiv m(\bmod 2m)}} (F(a-b) - F(a+b))\\ &= \sum_{\substack{(a+b)x=n\\x\equiv 0(\bmod m)}} (F(a-b) - F(a+b)) - \sum_{\substack{(a+b)x=n\\x\equiv 0(\bmod 2m)}} (F(a-b) - F(a+b))\\ &= -\sum_{k|\frac{n}{m}}(k-1)F(k) + \sum_{k|\frac{n}{2m}}(k-1)F(k).\end{aligned}$$

□

Proof of Theorem 1.6. Proof is similar to Theorem 1.4 except for letting $f(x) = x^{2l+1}$ in Lemma 3.6. □

Lemma 3.7. *Let $m, n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3 \\ (a+b)x=n \\ a\equiv 0(\bmod m) \\ b\equiv 0(\bmod m)}} (F(a-b) - F(a+b)) = -\sum_{k|\frac{n}{m}} (k-1)F(mk).$$

Proof. We have

$$\begin{aligned}
&\sum_{\substack{(a+b)x=n \\ a\equiv 0(\bmod m) \\ b\equiv 0(\bmod m)}} (F(a-b) - F(a+b)) \\
&= \sum_{(a+b)x=\frac{n}{m}} (F(ma-mb) - F(ma+mb)) \\
&= F(0) \sum_{\substack{(a+b)x=\frac{n}{m} \\ a=b}} 1 + \sum_{k\geq 1} F(k) \left\{ \sum_{\substack{(a+b)x=\frac{n}{m} \\ ma-mb=k}} 1 - \sum_{\substack{(a+b)x=\frac{n}{m} \\ ma-mb=-k}} 1 - \sum_{\substack{(a+b)x=\frac{n}{m} \\ ma+mb=k}} 1 \right\} \qquad (3.13) \\
&= F(0) \sum_{\substack{(a+b)x=\frac{n}{m} \\ a=b}} 1 - \sum_{k\geq 1} F(k) \sum_{\substack{(a+b)x=\frac{n}{m} \\ ma+mb=k}} 1.
\end{aligned}$$

Since $F(0) = 0$, Eq. (3.13) can be written as

$$\begin{aligned}
&\sum_{\substack{(a+b)x=n \\ a\equiv 0(\bmod m) \\ b\equiv 0(\bmod m)}} (F(a-b) - F(a+b)) = -\sum_{k\geq 1} F(k) \sum_{\substack{(a+b)x=\frac{n}{m} \\ ma+mb=k}} 1 \\
&= -\sum_{k\geq 1} F(k) \sum_{\substack{(a+b)x=\frac{n}{m} \\ a+b=\frac{k}{m}}} 1 = -\sum_{\substack{k\geq 2m \\ k|n \\ m|k}} F(k) \left(\frac{k}{m} - 1\right) = -\sum_{\substack{mk\geq 2m \\ mk|n}} F(mk)\,(k-1) \\
&= -\sum_{\substack{k\geq 2 \\ k|\frac{n}{m}}} F(mk)\,(k-1) = -\sum_{\substack{k\geq 1 \\ k|\frac{n}{m}}} (k-1)\,F(mk) = -\sum_{k|\frac{n}{m}} (k-1)\,F(mk).
\end{aligned}$$

□

Proof of Theorem 1.7. Proof is similar to Theorem 1.4 except for letting $f(x) = x^{2l+1}$ in Lemma 3.7. □

Lemma 3.8. *Let $m, n \in \mathbb{N}$. Let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,x)\in\mathbb{N}^3 \\ (a+b)x=n \\ a\equiv m(\bmod 2m) \\ b\equiv m(\bmod 2m)}} (F(a-b) - F(a+b)) = -\sum_{k|\frac{n}{2m}} kF(2mk).$$

Proof. Proof is similar to the above. We observe that

$$\begin{aligned}
&\sum_{\substack{(a+b)x=n\\ a\equiv m(\bmod 2m)\\ b\equiv m(\bmod 2m)}} (F(a-b)-F(a+b))\\
&=\sum_{\substack{(a+b)x=n\\ a\equiv -m(\bmod 2m)\\ b\equiv -m(\bmod 2m)}} (F(a-b)-F(a+b))\\
&=\sum_{(2ma+2mb-2m)x=n} (F(2ma-2mb)-F(2ma+2mb-2m))\\
&=\sum_{(a+b-1)x=\frac{n}{2m}} (F(2ma-2mb)-F(2ma+2mb-2m)) \qquad (3.14)\\
&=\sum_{k\geq 1}F(k)\left\{\sum_{\substack{(a+b-1)x=\frac{n}{2m}\\ 2ma-2mb=k}}1-\sum_{\substack{(a+b-1)x=\frac{n}{2m}\\ 2ma-2mb=-k}}1-\sum_{\substack{(a+b-1)x=\frac{n}{2m}\\ 2ma+2mb-2m=k}}1\right\}\\
&=-\sum_{k\geq 1}F(k)\sum_{\substack{(a+b-1)x=\frac{n}{2m}\\ 2ma+2mb-2m=k}}1
\end{aligned}$$

Thus (3.14) is

$$\begin{aligned}
&\sum_{\substack{(a+b)x=n\\ a\equiv -m(\bmod 2m)\\ b\equiv -m(\bmod 2m)}} (F(a-b)-F(a+b)) = -\sum_{k\geq 1}F(k)\sum_{\substack{(a+b-1)x=\frac{n}{2m}\\ a+b-1=\frac{k}{2m}}}1\\
&=-\sum_{\substack{k\geq 2m\\ k|n\\ 2m|k}}F(k)\frac{k}{2m}=-\sum_{\substack{2mk\geq 2m\\ 2mk|n}}F(2mk)k=-\sum_{\substack{k\geq 1\\ k|\frac{n}{2m}}}kF(2mk)\\
&=-\sum_{k|\frac{n}{2m}}kF(2mk).
\end{aligned}$$

□

Proof of Theorem 1.8. Proof is similar to Theorem 1.4 except for letting $f(x)=x^{2l+1}$ in Lemma 3.8. □

4 Conclusions

Mainly, we consider the Liouville's Identity and the generalization of the Liouville's Identity by Huard, Ou, Spearman and Williams, which are enable us to deduce some combinatoric convolution sums formulae as

$$\sum_{i=0}^{l}\binom{2l+1}{2i+1}\sum_{\substack{(a,b,x)\in\mathbb{N}^3\\(a+b)x=n\\x\equiv m(\bmod 2m)}}a^{2l-2i}b^{2i+1}$$
$$=\frac{1}{2}\left\{\sigma_{2l+2}(\frac{n}{m})-\sigma_{2l+1}(\frac{n}{m})-\sigma_{2l+2}(\frac{n}{2m})+\sigma_{2l+1}(\frac{n}{2m})\right\},$$
$$\sum_{i=0}^{l}\binom{2l+1}{2i+1}\sum_{\substack{(a,b,x)\in\mathbb{N}^3\\(a+b)x=n\\a\equiv m(\bmod 2m)\\b\equiv m(\bmod 2m)}}a^{2l-2i}b^{2i+1}=\frac{(2m)^{2l+1}}{2}\sigma_{2l+2}(\frac{n}{2m}),$$

(see Theorem 1.6 and Theorem 1.8) and etc., by dealing with an odd function for $m, n \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{0\}$.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Berndt BC. Number Theory in the Spirit of Ramanujan. Amer. Math. Soc., Providence, Rhode Island, USA; 2006.

[2] Glaisher JWL. On the square of the series in which the coefficients are the sums of the divisors of the exponents. Mess. Math. 1884;14:156-163.

[3] Glaisher JWL. On certain sums of products of quantities depending upon the divisors of a number. Mess. Math. 1885;15:1-20.

[4] Glaisher JWL. Expressions for the first five powers of the series in which the coefficients are sums of the divisors of the exponents. Mess. Math. 1885;15:33-36.

[5] Berndt BC. Ramanujan's Notebooks. Part II. Springer-Verlag, New York; 1989.

[6] Hahn H. Convolution sums of some functions on divisors. Rocky Mountain J. Math. 2007;37:1593-1622.

[7] Huard JG, Ou ZM, Spearman BK, Williams KS. Elementary Evaluation of Certain Convolution Sums Involving Divisor Functions. Number theory for the millennium, II. 2002;229-274.

[8] Melfi G. On Some Modular Identities. de Gruyter, Berlin. 1998;371-382.

[9] Cho B, Kim D, Koo JK. Divisor functions arising from q-series. Publ. Math. Debrecen. 2010;76:495-508.

[10] Cho B, Kim D, Koo JK. Modular forms arising from divisor functions. J. Math. Anal. Appl. 2009;356:537-547.

[11] Alaca A, Alaca S, Williams KS. The convolution sum $\sum_{m<n/16}\sigma(m)\sigma(n-16m)$. Canad. Math. Bull. 2008;51:3-14.

[12] Alaca A, Alaca S, Williams KS. The convolution sums $\sum_{l+24m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+8m=n} \sigma(l)\sigma(m)$. M. J. Okayama Univ. 2007;49:93-111.

[13] Williams KS. Number Theory in the Spirit of Liouville. London Mathematical Society, Student Texts 76, Cambridge; 2011.

[14] Liouville J. Sur quelques formules générales quipeuvent être utiles dans la théorie des nombres. 1858;3:143-152.

[15] Liouville J. Sur quelques formules générales quipeuvent être utiles dans la théorie des nombres. 1858;3:241-250.

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