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General Equilibria

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Abstract

This research work is devoted to general equilibria in abstract spaces, through the agency of the ideal points, fixed points of multifunctions, Isac's cones, Choquet boundaries and applied mathematics for academic and industrialists engaged in eco - efficiency. It can be also considered as a short survey of the recent results obtained in this field.

Keywords: Ideal point, multifunction, Isac's cone, Choquet boundaries.

1 Introduction

There exists a great variety of definitions for the equilibria in: Chemistry, Economics, Physics, Life Sciences and Allied Applications like Physiology, Medicine, Politics and so on. But what it means in Mathematics? In this research work we present properties of the equilibria (ideal or critical) points for the most important class of generalized dynamical systems through the agency of the connections with the Efficiency and Choquet boundaries.Thus, Section 2 is devoted to the study of the existence and the properties for the equilibrium points sets. In Section 3 two important coincidence results between the (approximate) equilibrium points sets and the Choquet boundaries are given. Section 4 comprises a new approximate modality for the equilibrium points sets and the last section includes two applications. In order to simplify the basic references we refer the reader only to [1–3].

2 Existence of Equilibria Points under Completeness

In this section we study the existence of critical points for a class of generalized dynamical systems in separated locally convex spaces ordered by (weak) supernormal cones, introduced by G. Isac in 1983 using the weak completeness. We note that the weakly complete cones are very important in functional analysis and potential theory.

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Let X be a real Hausdorff locally convex space with the topology induced by a family $P=\{P_\alpha:\alpha\in I\}$ of seminorms, ordered by a convex cone K and its topological dual space X^*.

Definition 2.1 *K is called Isac's (nuclear or supernormal) cone if for every $p\in P$ their exists $f\in X^*$ such that $p(x)\le f(x)$ for all $x\in K$.*

Many examples of such cones as these, their importance for Pareto efficiency and full nuclearity are exhibited in and so on.

Let A be a non-empty set in a Hausdorff locally convex space X ordered by a convex cone K.

Definition 2.2. *A set valued map $\Gamma: A\to 2^A$ is called a generalized dynamical system if $\Gamma(a)$ is non-empty for every $a\in A$.*

Let $\Gamma: A\to 2^A$ be the generalized dynamical system defined by $\Gamma(a)=A\bigcap(a-K), a\in A$.

Definition 2.3. *We say that a_0 is an equilibrium (ideal or critical point) for Γ or a minimal efficient (Pareto minimum) point for A with respect to K, in notation, $a_0\in MIN_K(A)$ if it satisfies one of the following equivalent conditions;*

(i) $A\bigcap(a_0-K)\subseteq a_0+K,$

(ii) $K\bigcap(a_0-A)\subseteq -K,$

(iii) $(A+K)\bigcap(a_0-K)\subseteq a_0+K,$

(iv) $K\bigcap(a_0-A-K)\subseteq -K.$

The *maximal efficient points* are defined similarly by replacing the convex cone K with $-K$. For this reason, in the sequel we consider the equilibrium points given by the efficient points as minimal. The term of efficient point will be frequently used instead of the equilibrium point in order to support the strong connection with the efficiency.

We recall that every supernormal cone is pointed, that is, $K\bigcap(-K)=\{a_0\}$ and, in any such a case as this, $a_0\in MIN_K(A)$ if and only if $A\bigcap(a_0-K)=\{a_0\}$, or equivalently $K\bigcap(a_0-K)=\{0\}$.

The problem consists to apply the mathematical modelling and the corresponding numerical simulation in this way to the real processes.

Theorem 2.1. *If $(X,P=\{p_\alpha:\alpha\in I\})$ is a separated locally convex space with the topology generated by a family P of seminorms and A is a non-empty complete set in X such that for every $p_\alpha\in P$ there exists a lower semicontinuous function $\varphi_\alpha: A\to R_+$ with $p_\alpha(x-y)\le\varphi_\alpha(x)-\varphi_\alpha(y), \forall x\in A, y\in\Gamma(x)$, then $MIN_K\neq\phi$.*

The importance of Isac's cones for the existence and the domination property of the equilibrium points is also illustrated by the following results.

Theorem 2.2. *Let* $A \subseteq B \subseteq A + K$. *If* K *is supernormal and* $B \cap (A_0 - K)$ *is bounded and complete for some non-empty set* $A_0 \subseteq A$, *then* $MIN_K(A) \neq \phi$.

Corollary 2.2.1. *Let* $A \subseteq B \subseteq A + K$. *If* K *is weakly supernormal and* $B \cap (A_0 - K)$ *is bounded and weakly complete for some non-empty set* $A_0 \subseteq A$, *then* $MIN_K(A) \neq \phi$.

In particular, if K *is a weakly supernormal cone in* X *and* $A \cap (a - K)$ *or* $(A + K) \cap (a - K)$ *is bounded and weakly complete for some* $a \in A$, *then* $MIN_K(A) \neq \phi$. *When the boundedness and weak completeness properties hold for every* $a \in A$, *then we have the following domination property* $A \subseteq MIN_K(A) + K$.

Since in every separated locally convex space any normal cone is weakly supernormal, the conclusion of the above corollary remains valid whenever K is normal. Using this remark one obtains the next existence results for efficient points.

Corollary 2.2.2. $MIN_K(A) \neq \phi$ *if one satisfies one of the following conditions;*

iii) (i) K *is closed, normal, weakly complete and* A *is weakly closed such that* $A \cap (a - K)$ *is bounded for some* $a \in A$. *We also have* $A \subseteq MIN_K(A) - K$ *if, under the above hypotheses,* $A \cap (a - K)$ *in bounded for every* $a \in A$,

iiii) (ii) K *is closed, normal and* A *is bounded and weakly complete. The domination property holds again,*

iiiii) K *is closed, normal, weakly complete and* $A + K$ *is weakly closed such that* $(A + K) \cap (a - K)$ *is bounded for some* $a \in A$.

The domination property $A \subseteq MIN_K(A) + K$ *holds if, in addition* $(A + K) \cap (a - K)$ *is bounded for any* $a \in A$.

Remark. 2.1. It is clear that all the above results remain valid if one replaces the hypothesis of normality on K with the weak normality.

Corollary 2.2.3. *If* A *is a non-empty, bounded and closed subset of* X *and* K *is well based (that is, generated by a non-empty convex, bounded set which does not contain the origin in its closure) by a complete set, then* $MIN_K(A) \neq \phi$ *and* $A \subseteq MIN_K(A) + K$.

Corollary 2.2.4. *If* A *is a non-empty, bounded and closed subset of a Banach space ordered by a convex cone* K *well based by a closed set, then* $MIN_K(A) \neq \phi$ *and* $A \subseteq MIN_K(A) + K$.

Remark 2.2. Since in a normed space a convex cone is supernormal if and only if it is well based, the last corollary offers some conclusion whenever K is supernormal. Moreover, the existence results given in this section show also the possibility to use the weakly complete cones for the study of the equilibrium (Pareto type optimization) in separated locally convex spaces. For comprehensive bibliography, we refer [6] and references therein.

3 Coincidence Results between Equilibria Points Sets, Choquet Boundaries and Related Topics

In our all futher considerations we suppose that (E,τ) is a Hausdorff locally convex space, where τ denotes its topology, K is a closed, convex pointed cone in E and ε is an arbitrary element of $K\setminus\{0\}$. On the vector space E we consider the usual order relation $\leq_K$ associated with K as follows: for $x,y\in E$ one defines $x\leq_K y$ iff $y\in x+K.$

Clearly, this order relation on E is closed, that is, the set G_K given by $G_K=\{(x,y)\in E\times E : x\leq_K y\}$ is a closed subset of $E\times E$. Also the set $G_{\varepsilon+K}=\{(x,y)\in E\times E : y\in x+\varepsilon+K\}$ is closed in $E\times E$ endowed with the usual product topology.

Definition 3.1. *If A is a non-empty subset of E, then $a_0\in A$ will be called an ε-critical point (or, ε- minimal element, ε- efficient point, Pareto ε- efficient point, ε- near to minimum point) of A with respect to K if there exists no $a_0\in A$ such that*

$$a_0-a-\varepsilon\in K,\ \textit{that is,}\ \ a_0-a-\varepsilon\in K\bigcap A=\phi.$$

The ε- critical points set of A with respect to K will be denoted by ε- $MIN_K(A)$ (ε- $eff(A,K)$).

Remark 3.1. It is clear that the concept of the ε- efficient points does not include the notion of efficient point,

$$MIN_K(A)\subseteq \varepsilon-MIN_K(A),\ \forall\varepsilon\in K\setminus\{0\} \text{ and } MIN_K(A)=\bigcap_{\varepsilon\in K\setminus\{0\}}\left[\varepsilon-MIN_K(A)\right].$$

Remark 3.2. A very interesting and important generalization of the approximative Pareto efficiency can be obtained by replacing ε with a non-empty subset of $K\setminus\{0\}$. In this way it is shown that the existence of this new type of efficient points for lower bounded sets characterizes the semi-Archimedian ordered vector spaces and the regular ordered locally convex spaces.

Definition 3.2 *A real function $f:E\to\mathbb{R}$ is caled $\varepsilon+K$ - increasing if $f(x_1)\geq f(x_2)$ whenever $x_1,x_2\in E$ and $x_1\in x_2+\varepsilon+K.$*

For a non-empty and compact subset X of E we recall some basic consideration in potential theory concerning the Choquet boundary of X with respect to a convex cone of continuous functions on X. Thus, we remember that if S is a convex cone of real continuous functions on X such that the constant function on X belong to S, it is min-stable (i.e., for every $f_1, f_2 \in S$ it follows $\inf(f_1, f_2) \in S$) and it separates the points of X, then on the set $M_+(X)$ of all positive Radon measures on X we associate the following order relation;

if $\mu, \nu \in M_+(X)$ then $\mu \leq s\, \nu$ iff $\mu(s) \leq \nu(s)$ for all $s \in S$. A measure $\mu \in M_+(X)$ is minimal with respect to the above order relation if for any continuous function $f : X \to \sqcup$ we have $\mu(Q_s f) = \mu(f)$, where $\mu(Q_s f) = \inf\{s \in S : f \leq s\}$. Particularly, if $x \in X$, then the Dirac measure ε_x is minimal iff $\varepsilon_x(Q_s f) = \varepsilon_x(f)$, that is, $Q_s f(x) = f(x)$ for every continuous function $f : X \to \sqcup$.

The set of all points $x \in X$ such that ε_x is minimal measure with respect to $\leq_s$ is named the Choquet boundary of X with respect to S and it is denoted by $\delta_s X$. Hence, if $C(X)$ is the usual Banach space of all real continuous functions on X, then

$$\delta_s X = \{x \in X : Q_s f(x) = f(x), \forall f \in C(X)\}.$$

A closed set $A \subseteq X$ is called S - absorbent if $x \in A$ and $\mu \leq_s \varepsilon_x$ implies $\mu(X \setminus A) = 0$. The trace on $\delta_s X$ of the topology on X in which the closed set coincides with X or with the absorbent subset of X contained in $\{x \in X : \exists\, s \in S$ with $s(x) < o\}$ is named usually the Choquet topology on $\delta_s X$.

An important connection between vector optimization and potential theory is the next coincidence of efficient points sets and Choquet boundaries in separated locally convex spaces which cannot be obtained as a consequence of the axiomatic potential theory.

Theorem 3.1. *$MIN_K(X)$ coincides with the Choquet boundary of X with respect to the convex cone of all real continuous functions which are increasing with respect to the order relation $\leq_K$. Consequently, the set $MIN_K(X)$ endowed with the trace topology τ_X induced on X by τ is a Baire space. Moreover, if X is metrizable, then $MIN_K(X)$ is a G_δ- set in (X, τ_X).*

Consequently,

Corollary 3.1.1

iii) *$MIN_K(X)$ and $MIN_K(X) \cap \{x \in X : s(x) \leq 0\}$ $(s \in S)$ are compact sets with respect to Choquet's topology;*

iiii) *$MIN_K(X)$ is a compact subset of X.*

Remark 3.3. In the conditions of Theorem 3.1 let us consider, on $MIN_K(X)$ endowed with the trace topology also denoted by τ_X, the following game between parteners A and B: each partener succesively chooses a non-empty set belonging to τ_X such that the player A makes the first choice and each player must choose a set in τ_X which should be included into the previous chosen set of the other player.

Let $G_1, G_1^{'}, G_2, G_2^{'}, ..., G_n, G_n^{'}, ...$ be the succesive options of the two players, $(G_1, G_2, ..., G_n, ...)$ represent the option expressed by A and $(G_1^{'}, G_2^{'}, ..., G_n^{'}, ...)$ the option made by B. One says that the player B wins if no matter the way A plays, he is able to make an option so that
$\bigcap_{x \in \square^*} G_n \neq \phi.$

Theorem 3.1 together with Choquet's results concerning the properties of the Choquet boundary shows that the above game on $MIN_K(X)$ is won by the player B.

Remark 3.4. Under the hypotheses of Theorem 3.1, the set $eff(X,K)$ coincides with the Choquet boundary of X only with respect to the convex cone of all real, continuous and K - increasing functions on X. Thus, for example, if X is a non-empty, compact and convex subset of E then, the Choquet boundary of X with respect to the convex cone of all real, continuous and concave functions on X coincides with the set of all extreme points for A, that is, with the set of elements $x \in X$ such that if $x_1, x_2 \in X, \lambda \in (0,1)$ and $x = \lambda x_1 + (1-\lambda)x_2$, then $x = x_1 = x_2$. But, it is easy that, even in infinite dimensional cases, an extreme points for a compact convex set is not necessary an efficient points and conversely.

The following result extends Theorem 3.1 for ε - effuciency.

Theorem 3.2. *If X is a non-empty subset of E, then the set $\varepsilon - eff(X,K)$ coincides with the Choquet boundary of X with respect to the convex cone all $\varepsilon + K$ - increasing real continuous functions on X. Consequently, the set $\varepsilon - eff(X,K)$ endowed with the trace topology is a Baire space and if (X, τ_X) is metrizable, then $\varepsilon - eff(X,K)$ is a G_δ - subset of X.*

4 A Generalized Modality for the Equilibria Points Sets

Let X be a vector space ordered by a convex cone K, K_1 a non-void subset of K and A a non-empty subset of X. The following definition introduces a new concept of (approximate) Pareto type efficient points which, particularly, leads to the well known notion of Pareto efficiency (in fact, the generalization in abstract spaces of the finite dimensional notion as we shall see in the next considerations.

Definition 4.1. *We say that* $a_0 \in A$ *is a* K_1-*equilibrium(* K_1-*Pareto minimal or efficient) point of* A, *in notation,* $a_0 \in eff\left(A,K,K_1\right)$ $\left(or\ a_0 \in MIN_{K+K_1}\left(A\right)\right)$ *if it satisfies one of the following equivalent conditions:*

(i) $A\bigcap\left(a_0-K-K_1\right)\subseteq a_0+K+K_1;$

(ii) $\left(K+K_1\right)\bigcap\left(a_0-A\right)\subseteq -K-K_1;$

In a similar manner one defines the Pareto (maximal) efficient points by replacing $K+K_1$ with $-\left(K+K_1\right)$.

Remark 4.1. $a_0 \in eff\left(A,K,K_1\right)$ iff it is a fixed point for the multifunction $F: A \to 2^A$ defined by $F\left(t\right)=\left\{a\in A: A\bigcap\left(a-K-K_1\right)\subseteq t+K+K_1\right\}$, that is, an *equilibrium point* for the generated generalized dynamical system

$\Gamma = F$.Consequently, for the existence of the Pareto type efficient points it can applied appropriate fixed point theorems concerning the multifunctions.

Remark 4.2. It is well known that whenever $K_1 \subset K\setminus\left\{0\right\}$, the existence of this new type of efficient points for lower bounded sets characterizes the semi Archimedian ordered vector spaces and the regular ordered locally convex spaces.

Remark 4.3. When K is pointed, that is, $K\bigcap\left(-K\right)=\left\{0\right\}$, then $a_0 \in eff\left(A,K,K_1\right)$ means that $A\bigcap\left(a_0-K-K_1\right)=\varnothing$ or, equivalently, $\left(K+K_1\right)\bigcap\left(a_0-A\right)=\varnothing$ for $0\notin K_1$ and $A\bigcap\left(a_0-K-K_1\right)=\left\{a_0\right\}$, respectively, if $0\in K_1$. Whenever K is pointed and $K_1=\left\{0\right\}$, from Definition 4.1, one obtains the well known usual notion of Pareto (minimal, efficient, optimal or admissible) point, abbreviated $a_0 \in eff\left(A,K\right)$ $\left(or\ a_0\in MIN_K\left(A\right)\right)$, that is, satisfying the next equivalent properties:

(i) $A\bigcap\left(a_0-K\right)=\left\{a_0\right\};$

(ii) $A\bigcap\left(a_0-K\setminus\left\{0\right\}\right)=\varnothing;$

(iii) $K\bigcap\left(a_0-A\right)=\left\{0\right\};$

(iv) $\left(K\setminus\left\{0\right\}\right)\bigcap\left(a_0-A\right)=\varnothing$

and we notice that

$$eff\left(A,K,K_1\right)=eff\left(A,K\right)=\bigcap_{\left\{0\right\}\neq K_2\subseteq K} eff\left(A,K,K_2\right).$$

It is clear that for any $\varepsilon \in K \setminus \{0\}$, taking $K_1 = \{\varepsilon\}$, it follows that $a_0 \in eff(A, K, K_1)$ if and only if $A \cap (a_0 - \varepsilon - K) = \emptyset$. In all these cases, the set $eff(A, K, K_1)$ was denoted by $\varepsilon - eff(A, K)$ (*or* $\varepsilon - MIN_K(A)$ and it is obvious that $eff(A, K) = \bigcap_{\varepsilon \in K \setminus \{0\}} [\varepsilon - eff(A, K)]$.

Remark 4.4. The following theorem offers the first important connection between the strong optimization and the (approximate) Pareto efficiency in the environment of ordered vector spaces, described initially on the previous Definition 4.1.

Theorem 4.1. *If we denote by* $S(A, K, K_1) = \{a_1 \in A : A \subseteq a_1 + K + K_1\}$ *and* $S(A, K, K_1) \neq \emptyset$, *then* $S(A, K, K_1) = eff(A, K, K_1)$.

Proof.

Clearly, $S(A, K, K_1) \subseteq eff(A, K, K_1)$.

Indeed, if $a_0 \in S(A, K, K_1)$ and $a \in A \cap (a_0 - K - K_1)$ are arbitrary elements, then $a \in a_0 + K + K_1$, that is, $a_0 \in eff(A, K, K_1)$, by virtue of (i) in Definition 2.1. Suppose now that $\overline{a} \in S(A, K, K_1) \neq \emptyset$ and there exists $a_0 \in eff(A, K, K_1) \setminus S(A, K, K_1)$.
From $\overline{a} \in S(A, K, K_1)$ it follows that $a_0 \in \overline{a} + K + K_1$, that is, $\overline{a} \in a_0 - K - K_1$, from which, since $\overline{a} \in A$ and $a_0 \in eff(A, K, K_1)$ we conclude that $\overline{a} \in a_0 + K + K_1$.
Therefore, $A \subseteq \overline{a} + K + K_1 \subseteq a_0 + K + K_1$, in contradiction with $a_0 \notin S(A, K, K_1)$ as claimed.

Remark 4.5. If $S(A, K, K_1) \neq \emptyset$, then $K + K_1 = K$ hence $eff(A, K, K_1) = eff(A, K)$. Indeed, let $a \in S(A, K, K_1)$. Then, $a \in a + K + K_1$ which implies that $0 \in K + K_1$.

Therefore, $K \subseteq K_1 + K + K = K_1 + K \subseteq K$. The above theorem shows that, for any non-empty subset of an arbitrary vector space, the set of all strong minimal elements with respect to any convex cone through the agency of every non-noid subset of it coincides with the corresponding set of Pareto (minimal) efficient points whenever there exists at least a strong minimal element, the result remaining obviously valid for the strong maximal elements and the Pareto maximal efficient points, respectively. Using this result and our abstract construction given in for the splines in H-locally convex spaces we concluded that the only best simultaneous and vectorial approximation for each element in the direct sum of a (closed) linear subspace and its orthogonal with respect to a linear (continuous) operator between two H-locally convex spaces is its spline function. We also note that it is possible to have $S(A, K, K_1) = \emptyset$ and $eff(A, K, K_1) = A$. Thus, for example, if one considers $X = R^n (n \in N, n \geq 2)$ endowed with the separated H-locally convex topology generated by the semi-norms $p_i : X \to R_+$, $p_i(x) = |x_i|$, $\forall\ x = (x_i) \in X$, $i = \overline{1, n}$, $K = R_+^n$, $K_1 = \{(0, ..., 0)\}$ and for each real number c we define

$A_c = \left\{ (x_i) \in X : \sum_{i=1}^{n} x_i = c \right\}$, then it is clear that $S\left(A_c, K, K_1\right)$ is empty and $eff\left(A_c, K, K_1\right) = A_c$.

In all our further considerations we suppose that X is a Hausdorff locally convex space having the topology induced by family $P = \{p_\alpha : \alpha \in I\}$ of seminorms, ordered by a convex cone K and its topological dual space X^*. In this framework, the next theorem contains a significant criterion for the existence of the approximative Pareto (minimal) efficient points, in particular, for the usual Pareto (minimal) efficient points, taking into account that the dual cone of K is defined by $K^* = \{x^* \in X^* : x^*(x) \geq 0, \forall x \in K\}$ and its attached polar cone is $K^0 = -K^*$. The version for the (approximative) Pareto (maximal) efficient points is straightforward.

Theorem 4.2. *If A is any non-empty subset of X and* K_1 *is every non-void subset of K, then* $a_0 \in eff\left(A, K, K_1\right)$ *whenever for each* $p_\alpha \in P$ *and there exists in the polar cone* K^0 *of K such that* $p_\alpha\left(a_0 - a\right) \leq x^*\left(a_0 - a\right) + \eta, \forall a \in A.$

Proof.
Let us suppose that, under the above hypotheses, $\left(K + K_1\right) \bigcap \left(a_0 - A\right) \not\subseteq -\left(K + K_1\right)$, that is, there exists $a \in A$ so that $a_0 - a \in K + K_1 \setminus \left(-K - K_1\right)$. Then, $a_0 - a \neq 0$ and, because X is separated in Hausdorff's sense, there exists $p_\alpha \in P$ such that $p_\alpha\left(a_0 - a\right) > 0$. On the other hand, there exists $n \in N^*$ sufficiently large with $p_\alpha\left(a_0 - a\right)/n \in (0,1)$ and the relation given by the hypothesis of theorem leads to $p_\alpha\left(a_0 - a\right) \leq x^*\left(a_0 - a\right) + p_\alpha\left(a_0 - a\right)/n$ with $x^* \in K^0$ and $n \to \infty$, which implies that $p_\alpha\left(a_0 - a\right) \leq 0$, a contradiction and the proof is completed.

Remark 4.6. In general, the converse of this theorem is not valid at least in (partially) ordered separated locally convex spaces as we can see from the example considered in the previous remark for n=2. Indeed, if one assumes the contrary in the corresponding, mathematical's background, then, taking $\eta = \frac{1}{4}$ it follows that for each $\lambda_0 \in [0,1]$ there exists $c_1, c_2 \leq 0$ such that

$$\left|\lambda_0 - \lambda\right| \leq \left(c_1 - c_2\right)\left(\lambda_0 - \lambda\right) + \frac{1}{4}, \forall \lambda \in [0,1].$$

Taking $\lambda_0 = \frac{1}{4}$ one obtains $\left|1 - 4\lambda\right| \leq \left(c_1 - c_2\right)\left(1 - 4\lambda\right) + 1, \forall \lambda \in [0,1]$ which for $\lambda = 0$ implies that $c_2 \leq c_1$ and for $\lambda = \frac{1}{2}$ leads to $c_1 \leq c_2$, that is, $\left|1 - 4\lambda\right| \leq 1, \forall \lambda \in [0,1]$, a contradiction.

Let us to consider for each function $\varphi : P \to K^* \setminus \{0\}$ the convex cone $K_\varphi = \{x \in X : p(x) \leq \varphi(p)(x), \forall p \in P\}$. The next theorem represents, in the more general context, a new important link between strong optimization and the approximative vector optimization together with its usual particular variant, respectively.

Theorem 4.3. *If there exists* $\varphi : P \to K^* \setminus \{0\}$ *with then*

$$eff(A,K,K_1) = \bigcup_{\substack{a\in A \\ \varphi\in P\to K^*\setminus\{0\}}} S\left(A \cap (a-K-K_1), K_\varphi\right)$$

for any non-empty subset K_1 *of K.*

Proof

If $a_0 \in eff(A,K,K_1)$ is an arbitrary element, then, in accordance with the point (i) of the Definition 4.1 and the hypothesis of the above theorem, we have $A \cap (a_0 - K - K_1) - a_0 \subseteq K + K_1 \subseteq K \subseteq K_\varphi$ for some $\varphi : P \to K^* \setminus \{0\}$.

Therefore, $a_0 \in S\left(A \cap (a_0 - K - K_1), K_\varphi\right)$.

Hence, $eff(A,K,K_1) \subseteq \bigcup_{\substack{a\in A \\ \varphi:P\to K^*\setminus\{0\}}} S\left(A \cap (a_0 - K - K_1), K_\varphi\right)$. Conversely, let now $a_1 \in S\left(A \cap (a_0 - K - K_1), K_\varphi\right)$ for at least one elements $a_0 \in A$ and $\varphi : P \to K^* \setminus \{0\}$. Then, $a_1 \in A \cap (a_0 - K - K_1)$ and $A \cap (a_0 - K - K_1) - a_1 \subseteq K_\varphi$, that is,
$p(a - a_1) \le \varphi(p)(a - a_1), \forall a \in A \cap (a_0 - K - K_1),\ p \in P$ which implies immediately that
$p(a_1 - a) \le -\varphi(p)(a_1 - a) + \eta, \forall a \in A \cap (a_0 - K - K_1),\ p \in P, \eta \in (0,1)$ and, by virtue of Theorem 2.2 one obtains $a_1 \in eff\left(A \cap (a_0 - K - K_1), K, K_1\right)$.

But $eff\left(A \cap (a_0 - K - K_1), K, K_1\right) \subseteq eff(A,K,K_1)$.

Indeed, for any $t \in eff\left(A \cap (a_0 - K - K_1), K, K_1\right)$ and $h \in A \cap (t - K - K_1)$ we have $h \in A \cap (a_0 - K - K_1) \cap (t - K - K_1) \subseteq t + K + K_1$, that is, $A \cap (t - K - K_1) \subseteq t + K + K_1$ and by point (i) of Definition 2.1 one obtains $t \in eff(A,K,K_1)$. This completes the proof.

Remark 4.7. The hypothesis $K \subseteq K_\varphi$ imposed upon the convex cone K is automatically satisfied whenever K is a supernormal (nuclear) cone and it was used only to prove the inclusion $eff(A,K,K_1) \subseteq \bigcup_{\substack{a\in A \\ \varphi:P\to K^*\setminus\{0\}}} S\left(A \cap (a - K - K_1), K_\varphi\right)$. When K is any pointed convex cone, A is a non-empty subset of X and $a_0 \in eff(A,K)$, then, by virtue of (i) in Definition 2.3, it follows that $A \cap (a_0 - K) = \{a_0\}$, that is, $A \cap (a_0 - K) - a_0 = \{0\} \subset K_\varphi$. Hence, $a_0 \in S\left(A \cap (a_0 - K), K_\varphi\right)$ for every mapping $\varphi : P : K^* \setminus \{0\}$ and the next corollary is valid.

Corollary 4.3.1. *For every non-empty subset A of any Hausdorff locally convex space ordered by an arbitrary, pointed convex cone K with its dual cone* K^* *we have*

$$eff(A,K) = \bigcup_{\substack{a\in A \\ \varphi:P\to K^*\setminus\{0\}}} S\left(A\cap(a-K),K_\varphi\right)$$

Remark 4.8. Clearly, the announced theorem represents a significant result concerning the possibilities of scalarization for the study of Pareto efficiency in separated locally convex spaces, as we can see also for the particular cases of Hausdorff locally convex spaces ordered by closed, pointed and normal cones. The above coincidence between the equilibria points sets and Choquet boundaries togeher with its immediate corrolaries remains valid if one replaces ε by K_1.

Finally, following [1] and [2], we present two studies concerning the eco – efficiency which were not used yet.

5 Some Applications

5.1 The Equilibria of Multidimensional Ecosystems

This part of the present section is devoted to some new considerations for the study of the equilibrium points in any "usual" multidimensional ecosystem through the agency of the corresponding interdependences assumed between its parties, in the optimal background offered by the natural resources and under the ecological constraints and conversely. The equilibrium points are examined in the context given by a suitable sustainable development of ecosystem, based on the dynamic interactions between its components, in particular, the ecological and the social phenomenons and processes and more, the natural resources and the human needs, respectively. Thus, the foundations of this research paper consist in some new and very possible critical original ideas. Thus, first of all, we consider the useful modern concept of "sustainable development" and we propose its new corresponding extension concerning the general optimal integration through the agency of equilibrium for all the component parties in an every multidimensional ecosystem. The more general abstract efficiency has its immediate applications in the study of the equilibrium for multidimensional ecosystems because we think that our vectorial and, consequently, strong notion named *over all sustainable dimensionality* is the main consequence and generates the individual competitive and the mutual *sustainable targeting.* Our spline functions (splines) introduced in H-locally convex spaces backgrounds with their proper optimal interpolation properties applied in the corresponding particular cases of cardinal splines involved for the long – time decision – making processes whenever it is known their previous behaviours in the field, with appropriate regularities and one asks their evolution in the immediate future on a convenient time period. One of the reasons for this approach is our opinion that in any best (optimal) context of real (natural) sustainability the interactions between the ecological systems and the dynamics of the economical and social processes must run properly having the maximum degrees of regularity. Finally, we formulate some open problems and we give adequate concluding remarks. Concerning the sustainability and the equilibria, by our opinion, one of the way to examine the multidimensional ecosystems is the study of the interactions between their components based on the tendency by the individual objectives targeting continued by the corresponding generalization with the mutuals and afterwards the multiple sustainable targeting which, through the agency of the convergence for the interests, lead to the optimal solution (the equilibrium) named by us the *over all sustainable dimensionality.* Starting from the analysis of the existence and the behavior of the solutions which generate the equilibrium points for an ecosystem with two conflicting component parties under adequate hypotheses and following the modelling sustainable use of natural resources with applications by simulation to fishery and forestry, here we shall try to unify some of the basic results of the above research papers and so on, offering a proposal of generalization.

Let us consider an ecosystem in $n \geq 2$ conflicting parties $X_i \left(i = \overline{1,n}\right)$. If we denote by $f_i\left(t, x_j, x_i\right)$ $\left(i, j = \overline{1,n}\right)$ the growth rate of the quality of life for X_i with respect to X_j, then in partial accordance wih the usual agreements and following our ideas the mutual interactions witch lead to the equilibrium points, amongst X_i and Xj can be described explicitly in many cases by the following kind of general differential equations system:

$$(1) \qquad \dot{x}_i = f_i\left(t, x_j, x_i\right)x_i - r_{ij}\left(t\right),\ \mathrm{i,j} = \overline{1,n}$$

where x_i represents the level of life's quality for X_i, $r_{ij}\left(t\right)$ means the time–dependent harvest of the interaction $X_i \leftrightarrow X_j$ and f_i $\left(i = \overline{1,n}\right)$ satisfies proper conditions (see, for example, the particular bidimensional cases and the unidimensional basic models in the economic theory of resource management).

Whenever for $i = \overline{1,n}$ there exists x_i^* solution in (1) such $x_i^* \leq x_i$ for all possible x_i and $t \in \left[t_0, t_i\right)$ one says that x_i^* is a *solution* to achive the *individual competitive targeting*; if $\begin{cases} x_i^*\left(t\right) \leq x_i\left(t\right) \\ x_j^*\left(t\right) \leq x_j\left(t\right) \end{cases}$ $\left(i \neq j\right)$ for all admissible x_i and x_j and $t \in \left[t_0, t_{ij}\right)$, then we say that we have a *mutual sustainable development* based on a *reciprocal sustainable targeting*. Obviously, if $x_i^*\left(t\right) \leq x_i\left(t\right)$ for all possible allowed values of $x_i, i = \overline{1,n}$ and *t* sufficiently large, then it is about of *the strong sustainable development* grounded on *the corresponding global sustainable targeting*. If there exists a feasible solution $\widetilde{x}_i$ $\left(i = \overline{1,n}\right)$ of (1) such that it is not possible to have $\left(x_1\left(t\right), x_2\left(t\right), \ldots, x_n\left(t\right)\right) < \left(\widetilde{x}_1\left(t\right), \widetilde{x}_2\left(t\right), \ldots, \widetilde{x}_n\left(t\right)\right)$ for *t* sufficiently vast, then we call $\left(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\right)$ *the vector sustainable equilibrium* for (1).We must specify that all these concepts are introduced here under the main restriction that, from the dynamical point o view, the equilibrium was considered being moulded on every cyclicity submitted at least to the approximate periodical states of system's life.

Let us consider as a pertinent example the following bidimensional case offered by an ecosystem made up of two parties X_1 and X_2. Suppose that unit X_1 represents the pollutive party (in our case the power plant) and it will upset the vital equilibrium of an environmental system X_2 of high scenic interest, measured with reference to the quality life of the system itself. In our case, since it is extremely significant to know how systems develop in time, let's consider the case as a non autonomous one. Let $f_2(t, x_1, x_2)$ be the growth rate of the quality of life of system X_2 (measured by a quantity inversely proportional to the particulate fall-out) and $f_1(t, x_1, x_2)$ the growth rate of system X_1 (measured by the kwh generated).

Since the building up of a model does not consist only in specifying the mathematical formulation which holds it, but also in defining the hypothesis on which its existence is based, let's suppose that the interdependance between the conflicting systems be such as to satisfy the four following conditions:

(i) In the absence of system X_1, the level of the quality of life for system X_2 is restored as natural flow in a exponential way, at the rate $a_2 > 0$:

$$X_2' = a_2.$$

We are implying that for system X_2 the problem arises from the its proximity to system X_1 and whenever the latter is missing, the former would resume growth up to a level which would allow full use of natural available resources, by the disappearence of pollutive particulates. The term a_2 represents the growth coefficient of the quality of life or, otherwise, the reduction of the pollution level. Such a value is positive since in the absence of the pollutive cone, which runs over the specific area, the natural course would resume exponentially by integral curve.

$$x_2(t) = x_2^0 e^{a_2 t}.$$

(ii) In the absence of the environmental system X_2, system X_1 will develop its activity, by a logistic growth up to the level of maximum productive capacity, that is,

$$x_1' = a_1 x_1 - a_{11} x_1^2$$

with asymptote a_1 / a_{11}. We are admitting, in conclusion, that for system X_1 the problem lies in the proximity of system X_2 and, therefore, in the absence of the latter, the former will perform its activity up to the maximum level a_1 / a_{11}, with serious damages to the environment by integral curve

$$x_1(t) = \frac{a_1}{\left(\frac{a_1}{x_0} - a_{11}\right) e^{-a_1 t} + a_{11}}$$

with $x_0 < a_1 / a_{11}$. By projection of integral curves on axis x_1 we get the orbits, or trajectories, of the differential equation and the set P, structured by these projections, is called *space of the phases* of the equation; the arrows shows the direction by which the projected point draws its orbit by the increase of *t* (Fig. 1).

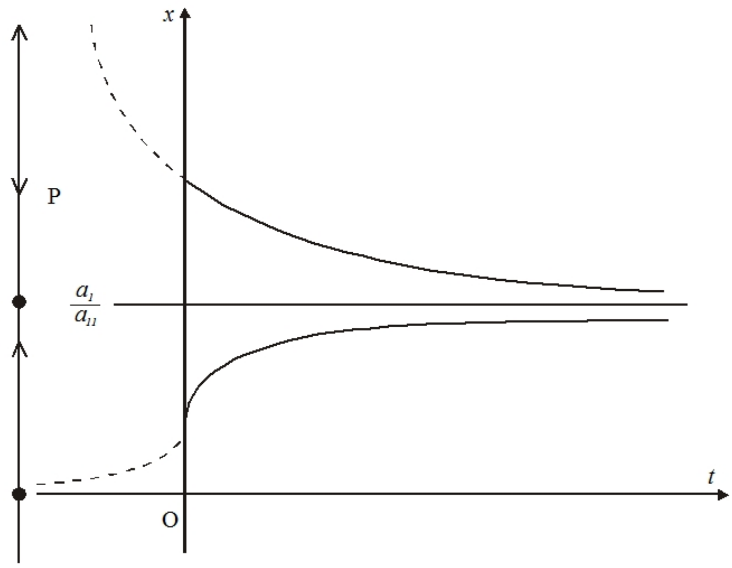

Fig. 1. The orbit by the increase of *t*.

(iii) The activity of system X_1 determines a reduction of the growth rate of the quality of life for system X_2 at a rate proportional to that activity and, more precisely, to the size and physical and chemical characteristics of the cone, which runs over the system X_2; let's suppose in this situation that the growth rate moves from a_2 to $\left(a_2 - a_{21}x_1\right)$, so that:

$$X_2' = \left(a_2 - a_{21}x_1\right)x_2$$
$$\frac{\partial f_2}{\partial x_1} < 0 = -a_{21}$$

On the meaning of coefficient a_{21} we assume that the interaction between the two systems is proportional both to X_1 and to $X_2, \alpha\left(t\right) x_1\, x_2$, with $\alpha\left(t\right)$ proportionally constant, as a measure of lower intensity of interaction. A fraction of this interaction turns out in the mitigation of the activity X_1, and it reduces the ill effects on the system X_2; as a first approximation, one could suppose that the size of this reduction is proportional to the intensity of the interaction at work. Specifying such a fraction as $\beta\alpha\left(t\right)x_1x_2$, where β measures the capacity of X_2 to protect from X_1, and since this is not possible, we can say that it measures the possibility by X_1 to mitigate its action over X_2, that is,

$a_{21} = \beta\alpha(t)$.

(iv) The reduction of the quality of life in system X_2 determines a decrease in the production rate of X_1 in proportion to the size of the reduction. Suppose that such a production rate $f_1(t, x_1, x_2)$ changes from the value $(a_1 - a_{11}\, x_2)$ to the value $(a_1 - a_{11}\, x_1 - a_{22}\, x_2)$, so that it becomes:

$$x_1' = (a_1 - a_{11}x_1 - a_{12}x_2)x_1 \text{ with } \frac{\partial f_1}{\partial x_2} < 0 = -a_{12} \text{ (Fig. 2)}$$

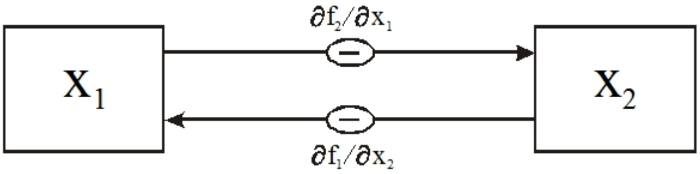

Fig. 2. The dependence between
X_1 and X_2

The coefficient a_{12} measures the disadvantage of system X_1 because of interaction with system X_2: particulate fall-out creates unpopular feelings towards the Electricity Board and, by specifying $a_{12} = \gamma a_{12}$, we can interpret γ as a measure of the pollution rate which forces X_1 to a decrease in its activity:

$$a_{21} = \gamma\beta\alpha(t).$$

In conclusion the interaction between the two systems is ruled by the following quadratic model, an equation system shaped as:

$$(2) \quad \begin{cases} x_1' = (a_1 - a_{11}x_1 - a_{12}x_2)x_1 \\ x_2' = (a_2 - a_{21}x_1)x_2 \end{cases}$$

The a_{11} coefficient can be called intersystem interaction coefficient, whilst the terms a_{12} and a_{21} can be called intrasystem interaction coefficients. The latter, moreover, could be envisaged as made up by two factors $a_{12} = cd_{12}$ and $a_{21} = cd_{21}$, where c measures the average level of interaction or interference between the two systems, whilst d_{12} and d_{21} measure the damage that such interaction involves for each system.

Having set out the problem in such a way, to study the interaction in the short run, we now proceed to discover, if there exist, the equilibrium solutions: a strong pollution at present, due also to adverse weather conditions, requires the search for a solution to be chosen amongst the feasible ones, such as to provoke an immediate reduction of the ill effects determined by the level of activity of X_1.

Defining D the determinant of the matrix of the equation in the above system which can be also called *an ecosystem matrix*, that is,

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{vmatrix} = -a_{12}a_{21} \text{ e con}$$

and by considering

$$H_1 = \begin{vmatrix} a_1 & a_{12} \\ a_2 & 0 \end{vmatrix} = -a_2a_{12},$$

$$H_2 = \begin{vmatrix} a_{11} & a_1 \\ a_{21} & a_2 \end{vmatrix} = a_2a_{11} - a_1a_{21}$$

then the equilibrium points of the mentioned equations system result as follows in terms of the following points:

$$E_0 : x_1 = 0, x_2 = 0;$$

$$E_1 : x_1 = \frac{a_1}{a_{11}}, x_2 = 0$$

$$E_2 : x_1 = \frac{a_2}{a_{21}} = \frac{H}{D}, x_2 = \frac{a_1a_{21} - a_{11}a_2}{a_{12}a_{21}} = \frac{H_2}{D}$$

By the usual analytic point of view the point E_1 is the intersection of straight line $x_1 = \frac{a_1}{a_{11}}$ with the axis x_1 ($x_2 = 0$) and the point E_2 is the intersection between the lines

$r_1 : x_1 - \frac{a_2}{a_{21}} = 0$ ad $r_2 : x_2 = \frac{a_1}{a_{12}} - \frac{a_{11}}{a_{12}} x_1$

geometric locus of the points where respectively x_2' and x_1' turn to null, beyond points on the axis $x_1 \left(x_2 = 0\right)$ for one and points of $x_2 \left(x_1 = 0\right)$ for the other one. The arrows show the signs of the derivatives: positive in the area in which it is concordant with the respective axis and negative in the area in which it is discordant, Fig. 3a and 3b. On the very straight lines, by definition, the orbit of the system by crossing them will become horizontally tangent on r_1 and vertically tangent on r_2. The point E_2 has always a positive abscissa, given that H_1 e D are both negative, whilst $x_2 \geq 0$ according to $H_2 = a_2a_{11} - a_1a_{21} \leq 0$, i.e. $\frac{a_2}{a_{21}} \leq \frac{a_1}{a_{11}}$:

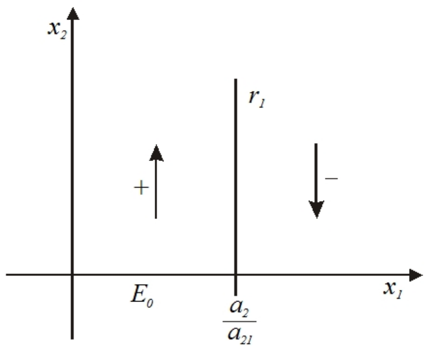

Fig. 3a

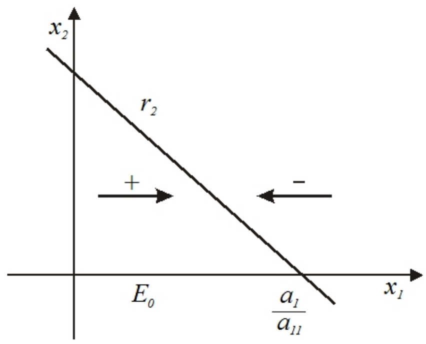

Fig. 3b

Fig. 3. Signs of the derivaties.

For a real understanding of the system considered, we are only interested in the quadrants where $x_1 > 0; x_2 < 0$ can yield some sense, as a sizeable fall-out of particulate generates worse quality of life.

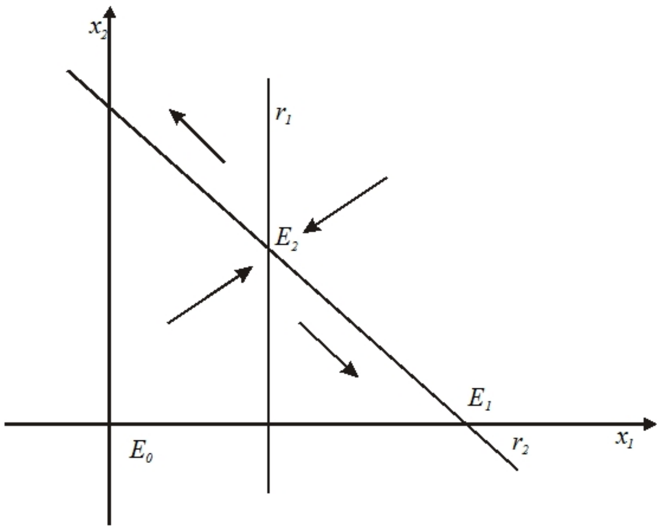

Fig. 4. Real understanding iof the system.

In Fig. 4 are drawn generic straight lines as the only way in which they can be set in the plane from a realistic point of view because of the sign of H_2, and the likely equilibrium points. Those points have no real meaning, when abscissa is greater than a_1/a_{11}, since that value is the optimal capacity of the environment for system X_1 (maximum yield in production corresponding to the asymptote of the logistic in accordance with the hypothesis (ii). Axis $x_1 > 0$ is constituted by four orbits:

$\{0\},]0, a_1/a_{11}, [, \{a_1/a_{11}\},]a_1/a_{11}, +\infty[,$

since if $x_2 = 0$, the first equation of the system (2) is reduced to equation in (i), whose orbits are just these sets; on the other hand axis x_2 is constituted by orbits:

$\{0\},]0, +\infty[.$

Therefore, if an orbit has a point in the first quadrant, it is all included there, so that points E_0, E_1, E_2 can not be spirals, or centers. Let's now move to analyze the nature of isolated equilibrium points, which have meaning for the ecosystem, in our case points E_1 ed E_2. To study their stability, let's refer to the linear approximation of system (2) in a point close to the equilibrium point considered. The Jacobian matrix of matrix of the is given by

$$J(x_1, x_2) = \begin{bmatrix} a_1 - 2a_{11}x_1 - a_{12}x_2 & -a_{12}x_1 \\ -a_{21}x_2 & a_2 - a_{21}x_1 \end{bmatrix}$$

As for point E_0 we have:

$$J(E_0) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

a matrix which has eigenvalues $\lambda_1 = a_1, \lambda_2 = a_2$. The origin, therefore, is an unstable node for linear approximation and, according to the theorem of the stability of solution of differential equation system, it is also stable for system (2)

The matrix calculated in E_1 becomes:

$$J(E_1) = \begin{bmatrix} -a_1 & -a_{12}\dfrac{a_1}{a_{11}} \\ 0 & a_2 - a_{21}\dfrac{a_1}{a_{11}} \end{bmatrix}$$

and has determinant $\neq 0$, which assures that the point is an isolated one, and has eigenvalues $\lambda_1 = -a_1, \lambda_2 = a_2 - a_{21}\dfrac{a_1}{a_{11}}$, that is if $a_2/a_{21} < a_1/a_{11}$ (H_2 as said <0) it is a stable node.

The matrix calculated in E_2 becomes:

$$J(E_2) = \begin{bmatrix} a_1 - 2a_{11}\dfrac{H_1}{D} - a_{12}\dfrac{H_2}{D} & -a_{12}\dfrac{H_1}{D} \\ -a_{21}\dfrac{H_2}{D} & a_2 - a_{21}\dfrac{H_1}{D} \end{bmatrix} = -\frac{1}{D}\begin{bmatrix} a_{11}H_1 & a_{12}H_1 \\ a_{21}H_2 & 0 \end{bmatrix}$$

it has also determinant $\neq 0$ (E_2 isolated point) and has characteristic equation:

$$\lambda^2 - (a_{11}H_1)\lambda + (-a_{12}a_{21}H_1H_2) = 0.$$

Being $H_2 < 0,$ it will have $\Delta > 0$ and, therefore, $\lambda_{1,2}$ real eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\left(a_{11}H_1 \pm \sqrt{(a_{11}H_1)^2 - 4DH_1H_2}\right),$$

resulting one positive and one negative.

Following the above mentioned theorem on stability, E_2 is a saddle point, an unstable equilibrium one, with identical configuration of orbits to that of the original system, meaning that they are not geometrically identical, but that the equilibrium point is the same. We are now able to draw the frame of phases for the system in the case Fig. 5. In this case, according to the fact that the initial point is above or below the parting lines of the saddle, labeled the system or will result overwhelming, respectively, in the interaction.

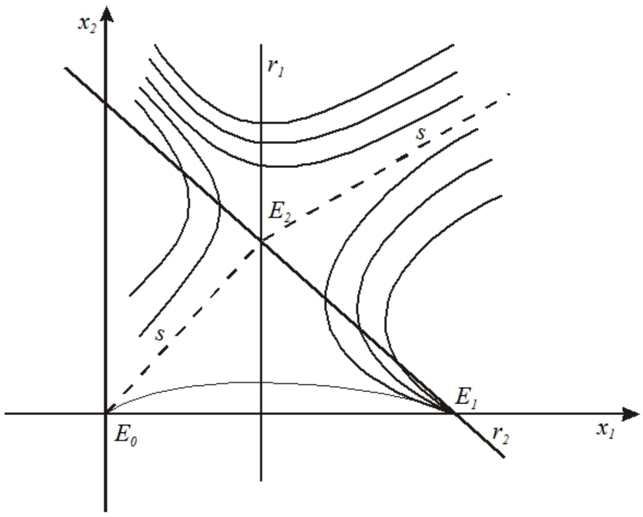

Fig. 5. Configuration of orbits.

It is obvious that all the above results can be applied in the study of multicriteria optimization problems with the objective maps taking values in Hausdorff locally convex spaces ordered by supernormal cones, in particular, in any normed linear (Banach) space ordered by a well based convex cone, usually in any Euclidian space ordered by customary positive pointed and cone. After all, any ecosystem is multidimensional al least with respect to the knowledge and from the mathematical point of view we appreciate that it can be tackled as a generalized dynamical system for which one looks up the (possibly approximate) equilibrium points in the following manner. Indeed, if we denote by A the set of all possible points $(x_1(t), x_2(t), ..., x_n(t))$ for the system (1), we consider $K = R_+^n$, and the generalized dynamical system Γ_0 given by $\Gamma_0(a) = A \cap (a - K)$, then any equilibrium point for Γ_0 represents an equilibrium point for (1). If we wish to consider thoroughly the study, then we can take $\Gamma_1(a) = A \cap [a - \text{int}(K)]$ or $\Gamma_\varepsilon(a) = A \cap (a - \varepsilon - K)$ whenever $\varepsilon \in R_+^n \setminus \{0\}$ and $a \in A$ or, more $\Gamma_T(a) = A \cap (a - T - K)$ respectively $\Gamma_T^0(a) = A \cap [a - T - \text{int}(K)]$ with $\phi \neq T \subseteq K \setminus \{0_K\}$.

In the second last approximate special case we remark immediately the important difference that, if we denote by $\varepsilon - MIN_K(A) = \{a_0 \in A : A \cap (a_0 - \varepsilon - K) = \phi\}$, then $MIN_K(A) = \bigcap_{\varepsilon \in K \setminus \{0\}} [\varepsilon - MIN_K(A)]$ with the similar corresponding equality for ε - maximum points which also may represent a sort of equilibrium points under appropriate conditions submitted also to stability.

It is clear that if one denotes by $\alpha_i = \alpha_i(t)$ the weight attached on x_i $(i = \overline{1, n})$ in (1), then any *strong minimum* for the function $f = f(t) = \sum_{i=1}^{n} \alpha_i \cdot x_i$ is in $MIN_K(A)$ whenever at least one α_i is not zero. This performs another modality to approach the equilibrium of any (multidimensional) ecosystem, that is, using the scalarization methods. Concerning the decision – making multidimensional ecosystems approached by the splines in H-locally convex spaces, let $(X, P = \{p_\alpha : \alpha \in I\})$ be a H–locally convex space, that is, a separated locally convex space with the seminorms satisfying the next usual parallelogram law:

$$p_\alpha^2(x + y) + p_\alpha^2(x - y) = 2[p_\alpha^2(x) + p_\alpha^2(y)], \forall x, y \in X, p_\alpha \in P$$

Let us consider M be a closed linear subspace of X for which there exists a H-locally convex space $(Y, Q = \{q_\alpha : \alpha \in I\})$ and a linear (continuous) operator $U : X \to Y$ such that $M = \{x \in X : (x, y)_\alpha = \langle Ux, Uy \rangle_\alpha, \forall y \in X, \alpha \in I\}$ where $(\cdot,\cdot)_\alpha$ $(\alpha \in I)$ denotes the scalar semiproduct which generates the seminorm $p_\alpha \in P$ and $\langle \cdot,\cdot \rangle_\alpha$ is the scalar semiproduct generating the seminorm $q_\alpha \in Q, \alpha \in I$.*The linear subspace of spline functions with respect to* U defined for the first time by us as the U-orthogonal of M is $M^\perp = \{x \in X : \langle Ux, U\zeta \rangle_\alpha = 0, \forall \zeta \in M, \alpha \in I\}$

Clearly, $M^{\perp}$ is the orthogonal of *M* in the H-locally convex sense, that is,

$$M^{\perp} = \left\{x \in X : \langle x, y \rangle_{\alpha} = 0, \forall y \in M\right\}$$

The main result concerning the immediate important connections between the best approximation and the vectorial optimization is the following

Theorem 1.

(i) *if* $K = R_{+}^{I}$*, then for each* $s \in M^{\perp}$*, every* $\sigma \in M^{\perp}$ *is the only solution of the next vectorial optimization problem:*

$$MIN_K\left(\left\{\left(q_{\alpha}\left(U(\eta - s)\right)\right) : \eta \in M \oplus M^{\perp} \text{ and } \eta - \sigma \in M\right\}\right)$$

(ii) *for every* $x \in M \oplus M^{\perp}$ *its spline function (projection onto* $M^{\perp}$*)* s_x *is the only solution for each of the following vectorial optimization problems:*

$$MIN_K\left(\left\{\left(q_{\alpha}\left(U(\eta - x)\right)\right) : \eta \in M^{\perp}\right\}\right);$$
$$MIN_K\left(\left\{\left(p_{\alpha}(x - y)\right) : y \in M^{\perp}\right\}\right);$$
$$MIN_K\left(\left\{\left(q_{\alpha}(U_y)\right) : y - x \in M\right\}\right).$$

Example 1.

Let

$$X = \left\{f \in C^{m-1}(R) : f^{(m-1)} \text{ is locally absolutely continuous and } f^{(m)} \in L_{loc}^{2}(R)\right\},\ m \geq 1$$

endowed with the H-locally convex topology generated by the next scalar semiproducts:

$$(x, y)_k = \sum_{h=0}^{m-1}\left[x^{(h)}(k) \cdot y^{(h)}(k) + x^{(h)}(-k) \cdot y^{(h)}(-k)\right] + \int_{-k}^{k} x^{(m)}(t) \cdot y^{(m)}(t)dt,\ k = 0,1,2,...$$

and $Y = L_{loc}^{2}(R)$ equipped with the H-locally convex topology induced by the scalar semiproducts

$$\langle x, y \rangle_k = \int_{-k}^{k} x(t) \cdot y(t)dt,\ k = 0,1,2,...$$

If $U : X \to Y$ is the derivation operator of order *m,* then

$$M = \left\{x \in X : x^{(h)}(\nu) = 0, \forall h = \overline{0,m-1}, \nu \in Z\right\} \text{ and}$$

$$M^{\perp} = \left\{s \in X : \int_{-k}^{k} s^{(m)}(t)x^{(m)}(t)dt = 0, \forall x \in M, k = 0,1,2,...\right\} =$$

$= \{ s \in X : s_{/(\nu,\nu+1)} \text{ is a polynomial function of degree } 2m-1 \text{ at most}, \forall \nu \in Z \}$
Any spline function *S* such as this is defined by

$$S(x) = p(x) + \sum_{h=0}^{m-1} c_1^{(h)} (x-1)_+^{2m-1} + \sum_{h=0}^{2m-1} c_2^{(h)} (x-2)_+^{2m-1} + \ldots + \sum_{h=0}^{m-1} c_0^{(h)} (-x)_+^{2m-1} + \ldots$$

where $u_+ = \frac{|u| + u}{2}, \forall u \in R,\ p$ is a polynomial function of degree 2*m*-1 at most perfectly determined together with the coefficients $c_\nu^h \left(h = \overline{0, m-1}, \forall \nu \in Z \right)$ by the interpolation conditions.

Therefore, for every function $f \in X$ there exists an unique function denoted by $S_f \in M^\perp$ such that $S_f^{(h)}(\nu) = f^{(h)}(\nu), \forall h = \overline{0, m-1}, \nu \in Z$. Hence *M* and $M^\perp$ give an orthogonal decomposition for the space *X*, that is $X = M \oplus M^\perp$.

Example 2.

Let $X = \{ f \in C^{m-1}(R) : f^{(m-1)} \text{ is locally absolutely continuous and } f^{(m)} \in L^2(R) \}$
endowed with the H-locally convex topology induced by the scalar semiproducts
$(x, y)_\nu = x(\nu) y(\nu) + \int_R x^{(m)}(t) y^{(m)}(t) dt, \nu \in Z,\ Y = L^2(R)$ with the topology generated by the inner product $\langle x, y \rangle_\nu = \int_R x(t) y(t) dt$ and $U : X \to Y$ be as usually the derivation operator of order *m*.

Then

$M = \{ x \in X : x(\nu) = 0, \forall \nu \in Z \}$ and

$$M^\perp = \left\{ s \in X : \int_R x^{(m)}(t) s^{(m)}(t) dt = 0, \forall x \in M \right\}$$

As in Example 1, $M^\perp$ coincides with the class of all piecewiese polynomial functions of order *2m* (degree *2m-1* at most) having their knots at the integer points of the real axis.

Consequently, for every function $f \in X$, there exists an unique spline function $S_f \in M^\perp$ which achieves an optimal interpolation for *f* on the set *Z,* that is, S_f satisfies the equalities $S_f(\nu) = f(\nu)$ for every $\nu \in Z$ and it is the unique solution for each corresponding optimization problem contained in the Theorem 1.

Thus, $M \oplus M^{\perp} = X$, that is, *M* and $M^{\perp}$ give an orthogonal decomposition of the space *X*, $M^{\perp}$ being again simultaneous and vectorial proximinal with respect to the family of seminorms generated by the above scalar semiproducts. Now it is very clear that our abstract construction for the splines and the above examples suggest another possibility for the study and analysis of the decision–making long-time problems concerning the multidimensional ecosystems whenever it is possible to approximate or to know in fact their demeanour for the criteria and the constraints point of view, because for any objective functions we exhibited the possibility to assessed its subsequent behaviour by the corresponding spline functions, under reasonable degrees of regularity. Now we present some open problems, conclusions and remarks. Upon close examination, the on top considerations lead to the following open problems and the new possibilities to tackle the multidimensional ecosystem programs from the mathematical point of view – concerning the optimality, with the inherent typical open problems:

1. Convergence methods on the existence of the solutions and the specific stability for the equilibrium points sets in the most general models of ecosystems.
2. Stability and approximate stability for the evolution of the ecosystems.
3. Mathematical methods governing the life of ecosystems.
4. The Pareto type efficiency (with its corresponding particular) cases of strong or contrivance between strong and vector optimization) can be applied for the investigation of the multidimensional decision – making ecosystem problems (see Section 2 and Section 3 of the present research paper).
5. The abstract method generating, in particular, the cardinal spline functions presented in Example 1 and Example 2 are very useful to find optimal interpolation solutions especially for the long-time decision-making multidimensional ecosystems programs whenever it is known their previous and/or the present behaviour and one desires to check the further behaviours under permissible and suitable degrees of smoothness for the objective functions.
6. The vectorial or approximate optimality versions seems to be more adequated than the strong (simultaneous) most propitious solutions, but this will be seen in the further studies because the natural context suggest rather permanent time-optimal combinations between these modalities of study.
7. Every preliminary analysis can be very relevant and must take priority before implementing any model of multicriteria analysis or which aims at resolving conflictuality.

5.2 New Proposals for the Study of the Equilibria in the Fish Wars

This section is devoted to the study of some types of equilibrium, under the types interdependences between different species of fish and in the context of biological and dynamic interactions. Techniques from splines in H-locally convex spaces and applications and Pareto efficiency are suggested to be used for these investigations.

The main aim of this paper is to present some new possibilities to investigate the equilibria in the fish wars through the agency of our spline functions and the recent results on the efficiencythe important results on Pareto optimization in the general context of ordered Hausorff locally convex spaces.For the interactions of fish species and the related topics, let us assume that there are

$p \in N^*$ owners and each of whom can fish $n \in N^*$ fish species. If one denotes by $\left(x_{1t}, x_{2t}, ..., x_{nt}\right)$ the stocks of each kind of fish at the moment *t*, then it is known (see, for instance [4] and its references for n=p=2) that the general biological growth is described by

$x_{it+1} = f_i\left(x_{1t}, x_{2t}, ..., x_{nt}\right), i = \overline{1,n}$

under the total catch of each species given by $c_{it} = \sum_{j=1}^{p} c_{jit}, i = \overline{1,n}.$

Consequently, each period's population is characterized by

$$x_{it+1} = f_i\left(x_{1t} - c_{1t}, x_{2t} - c_{2t}, ..., x_{nt} - c_{nt}\right), i = \overline{1,n}$$

and if we suppose, in the natural background, that the utility from consuming the *n* species of fish is $u\left(c_{j1t}, c_{j2t}, ..., c_{jnt}\right), j = \overline{1,p}$, then every $j = \overline{1,p}$ of owners is interested to maximize the corresponding sum of the discounted utility, that is, to solve the following problem

$$(\mathrm{P_j}) \quad \underset{\left(c_{j1t}, c_{j2t}, ..., c_{jnt}\right)}{MAX} \sum_{t=1}^{\infty} \delta_j^t u\left(c_{j1t}, c_{j2t}, ..., c_{jnt}\right), 0 < \delta_j < 1, j = \overline{1,p}.$$

whenever $x_{it} \geq c_{it} \geq 0$ and $x_{it+1} = f_i\left(x_{1t} - c_{1t}, x_{2t} - c_{2t}, ..., x_{nt} - c_{nt}\right)$ for any $i = \overline{1,n}.$

In the next considerations, we propose new modalities for the research investigations of these problems for $p, n \geq 2$. The above spline functions and efficiency can be applied in the fish wars to be analysed like multidimensional ecosystems as follows: first of all, we remark that, taking into account known values for the objective functions (and for some consecutive derivatives when these exist) defined by

$$\varphi_j\left(t\right) = \sum_{s=1}^{t} \delta_j^s u\left(c_{j1s}, c_{j2s}, ..., c_{jns}\right), j = \overline{1,p}, t \in \left[1, +\infty\right)$$

during a previous time period it is possible to use our splines in order to approximate and optimal interpolate them. It is clear that this procedure depends on the degrees of regularity for any objective function $\varphi_j, j = \overline{1,p}$ and the adequate restrictions. Afterwards, one can applied the strong individual or the strong global maximization, respectively to solve the problem $\left(P\right) = \bigcap_{j=1}^{p} \left(P_j\right)..$

Also, it is obvious that any strong global maximization solution concerning the equilibrium is a strong individual maximization equilibrium. From the vector optimization point of view which we consider more appropriate for the "reconciliation" of the real factors the global problem can be studied from the efficiency point of view:

if we denote by A the set of all

$$\varphi = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_p(t)) \quad \text{or}$$

$$\left(\sum_{t=1}^{\infty} \delta_1^t \cdot u(c_{11t}, c_{12t}, \ldots, c_{1nt}), \sum_{t=1}^{\infty} \delta_2^t \cdot u(c_{21t}, c_{22t}, \ldots, c_{2nt}), \ldots, \sum_{t=1}^{\infty} \delta_p^t \cdot u(c_{p1t}, c_{p2t}, \ldots, c_{pnt})\right)$$

respectively, we consider $K = R_+^p$ and the generalized dynamical system $\Gamma_0 : A \to 2^A$ given by $\Gamma_0(a) = A \cap (a + K)$, then any (approximate) equilibrium point for Γ_0, represents an (approximate) equilibrium for our problem $(P_1, P_2, \ldots, P_p)$.

If we wish to continue the study, then we can take $\Gamma_1(a) = A \cap [a + \text{int}(K)]$ or $\Gamma_T(a) = A \cap (a + T + K)$ respectively $\Gamma_T^0 = A \cap [a - T - \text{int}(K)]$ with $\varnothing \neq T \subseteq K$. For $T = \{\varepsilon\}$ and $\varepsilon \in K \setminus \{0_K\}$ taking $\varepsilon - \mathit{eff}(A_1 - K) = \{a_0 \in A : A \cap (a_0 + \varepsilon + K) = \varnothing\}$ one obtains $\mathit{eff}(A, -K) = \bigcap_{\varepsilon \in K \setminus \{0_K\}} [\varepsilon - \mathit{eff}(A, -K)]$, that is, another posibility to approximate the vectorial equilibria points.

6 Conclusion

This research work represents an original scientific contribution for the study of general equilibria, It can be developed with appropriate numerical methods for mathematical programming concerning efficiency, optimization and conversely.

Competing Interests

Author has declared that no competing interests exist.

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