British Journal of Mathematics & Computer Science
6(3): 172-186, 2015, Article no.BJMCS.2015.071
ISSN: 2231-0851

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A New Chebyshev Spectral-collocation Method for Solving a Class of One-dimensional Linear Parabolic Partial Integro-differential Equations

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Article Information
DOI: 10.9734/BJMCS/2015/14353
Editor(s):
(1) Jacek Dziok, Institute of Mathematics, University of Rzeszow, Poland.
(2) Paul Bracken, Department of Mathematics, University of Texas-Pan American, USA.
Reviewers:
(1) Anonymous, India. and (2) Suzan Jabbar Obaiys, Physics Department, Universiti Putra Malaysia, Malaysia. and (3) Anonymous, United Arab Emirates.
Complete Peer review History:
http://www.sciencedomain.org/review-history.php?iid=733&id=6&aid=7533

Original Research Article

Received: 27 September 2014
Accepted: 19 November 2014
Published: 31 December 2014

Abstract

In this work, the Chebyshev spectral-collocation method is applied to obtain approximate solution for some types of linear parabolic partial integro–differential equations (PPIDEs).
In the first approach, we convert our equation into two coupled Volterra integral equations of the second kind by using a proper transformation.
In the second approach, the integration in the resulting equations are approximated by replacing the integrand by its interpolating polynomials in terms of the Chebyshev polynomials instead of using the approximation by Gauss quadrature rules.
After approximation a linear algebraic system were raised, then it tested by the conditional number.
Finally, some numerical examples are included to illustrate the validity and applicability of the proposed technique.

Keywords: Spectral collocation method, Chebyshev polynomials; Chebyshev differentiation matrices; Lagrange basis function; parabolic partial integro–differential equations of Volterra type; Conditional number

2010 Mathematics Subject Classification: 35R09; 45K05; 97N50; 45D05

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1 Introduction

In recent years, the scientists have paid much attention to spectral method due to its high accuracy; they also have exponential rates of convergence (see for instance, [1], [2], [3], [4], [5], [6] and the references therein). Spectral methods were employed for numerical solution many types of integral and differential equations, due to their ease of using them for finite and infinite domains [7], [8], [9], [10]. Many problems in science and engineering arise in the mixed system of a partial differential equation and the integral terms involving the unknown function.

We are concerned with an efficient numerical approximation scheme of the mathematical model of a physical phenomenon involving contaminant transport process with memory term. Mathematically, the simplest example of such a problem would be represented by the one dimensional parabolic partial integro–differential equation of the form:

$$\partial_t u(x,t) - u_{xx}(x,t) + p(x)u_x(x,t) = \int_0^t k(x,t-s)u_{xx}(x,s)ds + f(x,t), \tag{1.1}$$

associated with the initial condition and Dirichlet boundary conditions:

$$u(x,0) = u_0(x), \qquad x \in \Omega = [-1,1], \tag{1.2}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad t \in I = [0,T]. \tag{1.3}$$

Here $(x,t) \in Q \equiv \Omega \times I$, where $I \equiv [0,T]$, $T > 0$ is a time interval and $\Omega \subseteq R \equiv [-1,1]$ is a bounded domain with boundary $\partial\Omega \equiv \{-1,1\}$. For implementation of high-order methods such as spectral methods, we assume that the functions $p(x), f(x,t), k(t-s)$ and $u_0(x)$ are smooth enough on Q.

Problems of type (1.1)–(1.3) represent the mathematical model of several physical phenomena involving process with memory term. For example, they describe poro-viscoelastic media [11], [12], heat conduction through materials with memory term [12], [13], the flows in porous media with nonlocal reactive [14], [15], and non-Fickian flow of fluid in porous media [16]. Many authors have considered the numerical solution of parabolic partial integro–differential equations by various methods. The method of lines was used in [17] and the discontinuous Galerkin method was reported in [18]. In [19], Galerkin mixed finite element methods and ADI orthogonal spline collocation have been investigated [20], [21] and the references therein. Also in [22], [23] some solutions to a class of (PPIDEs) have been introduced. In [24], [25], the method of radial basis functions was used to nonlinear and linear equations respectively. Finally, F. Fakhar-Izadi and M. Dehghan proposed spectral method for parabolic-type Volterra integro-differential equations based on Legendre collocation scheme [26].

As mentioned in [27], Spectral method with many types were devoted to Volterra integral and ordinary Volterra. In [28], H. Brunner has used Collocation methods for second-order Volterra integro-differential equations. Multistep collocation method is also used for Volterra integral equations [29]. Chebyshev spectral collocation method for the solution of Volterra integral and ordinary Volterra integro-differential equations are discussed in [30]. In [31], Tang introduces Legendre-spectral method with its error analysis for ordinary Volterra integro-differential equation of the second kind. Another spectral method using Legendre spectral Galerkin method was introduced for second-kind Volterra integral equations [32]. Also some authors [33], [34], [35] have introduced Chebyshev spectral–collocation method for Volterra integral equations. Most of the reported papers were specified to VIEs and ordinary VIDEs. However, in this article the solution of parabolic partial integro–differential ones (PPIDEs) were considered. In this article, the new Chebyshev spectral–collocation method was applied for both space and time that are an extension of the method presented in [34] and [35].

The organization of this work is as follows. In Section 2, some preliminaries, notations and the new Chebyshev spectral–collocation method introduce for numerically solving the described problem. In this section, the resulting set of algebraic linear equations are formed. In Section 3, presents the study the convergence behavior of the presented method and some numerical examples to demonstrate the effectiveness of the proposed method. Finally, in Section 4 we review some concluding remarks.

2 Chebyshev Spectral–Collocation Method

In this section, the Chebyshev spectral-collocation method is applied to problem (1.1)–(1.3). First some basic properties of the most commonly used set of orthogonal polynomials [36], [37]; Chebyshev polynomials will be introduced.

2.1 Chebyshev polynomials

The Chebyshev polynomials $T_n(x), n = 0, 1, ...$, are the Eigen-functions of the singular Sturm-Liouville problem [7], [6]

$$\frac{d}{dx}\left(\sqrt{1-x^2}\frac{dT_n(x)}{dx}\right) + \frac{n^2}{\sqrt{1-x^2}}T_n(x) = 0.$$

They are mutually orthogonal with respect to L_2-inner product on the interval $(-1, 1)$ with the weight function $\omega(x) = 1/\sqrt{1-x^2}$. This imply

$$\int_{-1}^{1} T_n(x)T_m(x)\omega(x)dx = \frac{c_n\pi}{2}\delta_{nm},$$

where δ_{nm} is the Kronecker delta, $c_0 = 2$ and $c_n = 1\,\forall\, n \geq 1$. The Chebyshev polynomials satisfy the following three-terms recurrence relation

$$\begin{aligned} T_0(x) &= 1, \quad T_1(x) = x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} T_0(x) &= T_1'(x), \quad 2T_1(x) = 0.5T_2'(x), \\ T_n(x) &= \frac{1}{2(n+1)}T_{n+1}'(x) - \frac{1}{2(n-1)}T_{n-1}'(x), \quad n \geq 1. \end{aligned} \tag{2.2}$$

A unique feature of the Chebyshev polynomials is their explicit relationship with a trigonometric function:

$$T_n(x) = \cos(n \arccos(x)). \tag{2.3}$$

In this work, the Chebyshev–Gauss (CG) points and the corresponding quadrature weights are taken as

$$x_n = -\cos\frac{(2n+1)\pi}{2N+2}, \quad \omega_n = \frac{\pi}{N+1}, \quad 0 \leq n \leq N, \tag{2.4}$$

in addition, the Chebyshev–Gauss–Lobatto (CGL) points and the corresponding quadrature weights are defined as

$$x_n = -\cos\left(\frac{n\pi}{N}\right), \quad \omega_n = \frac{\pi}{c_nN}, \quad c_n = \begin{cases} 2, & n = 0, N, \\ 1, & 1 \leq n \leq N-1. \end{cases} \tag{2.5}$$

2.2 Chebyshev Spectral–Collocation Scheme

Now, consider problem (1.1)–(1.3) on bounded domain Q. Because of the orthogonally property of the Chebyshev polynomials on the interval $[-1,1]$, we transfer equation (1.1) from $[0,T]$ to an equivalent problem defined in the interval $[-1,1]$, by using the next substitution [31], [38]

$$t = \frac{T}{2}(\tau+1), \qquad \tau \in [-1,1],$$

then equation (1.1) becomes, for all $x,\ \tau \in [-1,1]$

$$\frac{2}{T}\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial x^2} + p(x)U_x = \int_0^{\frac{T}{2}(\tau+1)} k\left(x, \frac{T}{2}(\tau+1) - s\right) u_{xx}(x,s)ds + g(x,\tau), \tag{2.6}$$

in which

$$U(x,\tau) := u\left(x, \frac{T}{2}(\tau+1)\right), \qquad g(x,\tau) := f\left(x, \frac{T}{2}(\tau+1)\right).$$

By using the following linear transformation in equation (2.6)

$$s = \frac{T}{2}(\rho+1), \qquad \rho \in [-1,1],$$

by converting the integration interval from $[0, T(\tau+1)/2]$ to the interval $[-1,\tau]$, so that equation (2.6) becomes

$$\frac{2}{T}\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial x^2} + p(x)U_x = \frac{T}{2}\int_{-1}^{\tau} K(x,\tau-\rho)U_{xx}(x,\rho)d\rho + g(x,\tau), \tag{2.7}$$

where $K(x,\tau-\rho) = k(x, T(\tau-\rho)/2)$. Define an auxiliary function [27], [38]

$$\Phi(x,\tau) = g(x,\tau) + \frac{T}{2}\int_{-1}^{\tau} K(x,\tau-\rho)U_{xx}(x,\rho)d\rho, \tag{2.8}$$

which will be used it in the next approximation. In order to approximate problem (1.1)–(1.3) by spectral methods, we rewrite equation (2.7) as two equivalent Volterra integro-differential equations by using equation (2.8) as follows

$$\begin{cases} \Phi(x,\tau) = g(x,\tau) + \dfrac{T}{2}\displaystyle\int_{-1}^{\tau} K(x,\tau-\rho)U_{xx}(x,\rho)d\rho, \\ \Phi(x,\tau) = \dfrac{2}{T}\dfrac{\partial U}{\partial \tau} - \dfrac{\partial^2 U}{\partial x^2} + p(x)U_x, \end{cases} \tag{2.9}$$

by integrating both sides in the second line of equation (2.9) over the interval $[-1,\tau]$ with respect to τ, we can get the required two linear equations of (VDEs) as the following

$$\begin{cases} \Phi(x,\tau) = g(x,\tau) + \dfrac{T}{2}\displaystyle\int_{-1}^{\tau} K(x,\tau-\rho)U_{xx}(x,\rho)d\rho, \\ U(x,\tau) = u_0(x) + \dfrac{T}{2}\displaystyle\int_{-1}^{\tau}\Big[\Phi(x,\xi) + U_{xx}(x,\xi) - p(x)U_x(x,\xi)\Big]d\xi. \end{cases} \tag{2.10}$$

Let the collocation points$\{(x_i,\tau_j)\}_{i,j}$ be the set of $(N+1)(M+1)$ points, in which $\{x_i\}_{i=0}^{N}$ are the Chebyshev–Gauss–Lobatto points (CGL nodes), and $\{\tau_j\}_{j=0}^{M}$ are the Chebyshev–Gauss points (CG nodes). Equation (2.10) holds at the collocation points such that, for all $0 \le j \le M$ yields

$$\begin{cases} \Phi(x_i,\tau_j) = g(x_i,\tau_j) + \dfrac{T}{2}\displaystyle\int_{-1}^{\tau_j} K(x_i,\tau_j-\rho)U_{xx}(x_i,\rho)d\rho, \quad 0 \le i \le N, \\ U(x_i,\tau_j) = u_0(x_i) + \dfrac{T}{2}\displaystyle\int_{-1}^{\tau_j}\Big[\Phi(x_i,\xi) + U_{xx}(x_i,\xi) - p(x_i)U_x(x_i,\xi)\Big]d\xi, \\ \hfill 1 \le i \le N-1. \end{cases} \tag{2.11}$$

For approximating the integral terms in equation (2.11), first the integral interval will be transfered from $[-1,\tau]$ to a fixed one $[-1,1]$ by using the following change of variables

$$\xi(\tau_j,\theta)=\rho(\tau_j,\theta)=\frac{\tau_j+1}{2}\theta+\frac{\tau_j-1}{2},\qquad \theta\in[-1,1],$$

where $\{\theta\}_{k=0}^{p}$are roots of the (p+1)–th Chebyshev polynomials. Then (2.11) becomes:

$$\begin{cases} \Phi(x_i,\tau_j)=g(x_i,\tau_j)+\dfrac{T(\tau_j+1)}{4} \\ \qquad\qquad .\displaystyle\int_{-1}^{1}K\Big(x_i,\tau_j-\rho(\tau_j,\theta)\Big)U_{xx}\Big(x_i,\tau_j-\rho(\tau_j,\theta)\Big)d\theta, \\ U(x_i,\tau_j)=u_0(x_i)+\dfrac{T(\tau_j+1)}{4}\displaystyle\int_{-1}^{1}\Phi\Big(x_i,\xi(\tau_j,\theta)\Big)d\theta+\dfrac{T(\tau_j+1)}{4} \\ \qquad\qquad .\displaystyle\int_{-1}^{1}\Big[U_{xx}\Big(x_i,\xi(\tau_j,\theta)\Big)-p(x_i)U_x\Big(x_i,\xi(\tau_j,\theta)\Big)\Big]d\theta. \end{cases} \tag{2.12}$$

Now, we use

$$\begin{cases} \Phi_N^M(x,\sigma)\approx\displaystyle\sum_{n=0}^{N}\sum_{m=0}^{M}l_n(x)F_m(\sigma)\Phi(x_n,\tau_m), \\ U_N^M(x,\sigma)\approx\displaystyle\sum_{n=0}^{N}\sum_{m=0}^{M}l_n(x)F_m(\sigma)U(x_n,\tau_m), \end{cases} \tag{2.13}$$

to approximate the functions Φ and U, where $l_n(x)$ and $F_m(\sigma)$are the n–th and m–th Lagrange basis functions corresponding to non–uniform meshes of $\{x_i\}$ and $\{\tau_j\}$ respectively. After enforcing the homogeneous boundary conditions at $x_0=-1$ and $x_N=1$ the first and the last terms in the interpolation polynomial of U are omitted. Therefore, we have

$$U_N^M(x,\sigma)\approx\sum_{n=1}^{N-1}\sum_{m=0}^{M}l_n(x)F_m(\sigma)U(x_n,\tau_m).$$

Now we can approximate U_x and U_{xx} in equation (2.12) by using the interpolation polynomial of U from the previous equation as the following

$$\begin{cases} (U_N^M)_x(x,\sigma)\approx\displaystyle\sum_{n=1}^{N-1}\sum_{m=0}^{M}l'_n(x)F_m(\sigma)U(x_n,\tau_m), \\ (U_N^M)_{xx}(x,\sigma)\approx\displaystyle\sum_{n=1}^{N-1}\sum_{m=0}^{M}l''_n(x)F_m(\sigma)U(x_n,\tau_m), \end{cases} \tag{2.14}$$

where $l'_n(x)$ and $l''_n(x)$ are the first and the second derivative of Lagrange interpolation basis polynomials $l_n(x)$ respectively, such that $l'_n(x)\in\mathbb{P}_{N-1}$ and $l''_n(x)\in\mathbb{P}_{N-2}$.

Now, the differentiation matrices can be defined as an essential building block for collocation methods used in spectral schemes [6], [7], [8]. Denoting $d_{i,n}^2:=l''_n(x_i)$, we introduce the second order differentiation matrix as the following $D^2=(d_{i,k}^2)_{0\le i,k\le N}$. By writing the entries of $D^2=D_{i,k}^2$ for $\{x_i\}$ as [39]:

$$D_{i,k}^2=\begin{cases} 2D_{i,k}\left(D_{i,i}-\dfrac{1}{x_i-x_k}\right), & i\neq k \\ -\displaystyle\sum_{l=0,i\neq l}^{N}D_{i,l}^2, & i=k, \end{cases}$$

where $D_{i,k}$ are the entries of the so–called differentiation matrix of dimension $(N+1)$. The entries of the differentiation matrix can be defined in [6] for (CGL) points as the following

$$D_{i,k} = \begin{cases} -\dfrac{2N^2+1}{6}, & i=k=0, \\ \dfrac{c_i}{c_k}\dfrac{(-1)^{i+k}}{x_i - x_k}, & i \neq k, \\ -\dfrac{x_i}{2(1-x_i^2)}, & 1 \le i = k \le N-1, \\ \dfrac{2N^2+1}{6}, & i=k=N, \end{cases}$$

with $c_i = 2$ for $i = 0, N$, where $c_i = 1$ otherwise.

In this paper, the integration in equation (2.12) can be approximated by replacing the integrad function by its interpolation polynomial approximation of Φ, U. In addition, by using the approximation to U_x and U_{xx} from equations (2.13) and (2.14), respectively in equation (2.12) and write

$$\begin{gathered} \Phi(x_i,\tau_j) = \Phi_{i,j}, \quad U(x_i,\tau_j) = U_{i,j}, \quad g(x_i,\tau_j) = g_{i,j}, \\ \rho(\tau_j,\theta_k) = \rho_{i,k}, \quad \xi(\tau_j,\theta_k) = \xi_{i,k}. \end{gathered}$$

Our main goal is to find the approximation $U_{i,j}$ so after replacing in equation (2.12) we get

$$\begin{cases} \Phi_{i,j} \approx g_{i,j} + \dfrac{T(\tau_j+1)}{4} \\ \qquad . \displaystyle\int_{-1}^{1} I_N^M \Big[K\Big(x_i, \tau_j - \rho(\tau_j,\theta)\Big)(U_N^M)_{xx}\Big(x_i, \tau_j - \rho(\tau_j,\theta)\Big)\Big] d\theta, \\ U_{i,j} \approx u_0(x_i) + \dfrac{T(\tau_j+1)}{4} \displaystyle\int_{-1}^{1} I_N^M \Big[\Phi_N^M\Big(x_i, \xi(\tau_j,\theta)\Big)\Big] d\theta + \dfrac{T(\tau_j+1)}{4} \\ \qquad . \displaystyle\int_{-1}^{1} I_N^M \Big[(U_N^M)_{xx}\Big(x_i, \xi(\tau_j,\theta)\Big) - p(x_i)(U_N^M)_x\Big(x_i, \xi(\tau_j,\theta)\Big)\Big] d\theta, \end{cases} \tag{2.15}$$

where I_N^M is the interpolation operator associated with the Chebyshev mesh points $\{x_i, \tau_j\}_{i,j}$ defined as the following

$$I_N^M Q(x,\tau) \approx \sum_{n=0}^{N}\sum_{m=0}^{M} l_n(x) F_m(\sigma) Q(x_n, \tau_m).$$

Now each equation in (2.15) can be reformulated respectively as

$$\begin{aligned} \Phi_{i,j} \approx g_{i,j} + \frac{T(\tau_j+1)}{4} \sum_{n=1}^{N-1}\sum_{m=0}^{M} l_n''(x_i) U(x_n,\tau_m) \\ . \sum_{z=1}^{p-1} l_z(x_i) \sum_{k=0}^{p} K(x_i, \tau_j - \rho_{j,k}) F_m(\rho_{j,k}) \int_{-1}^{1} F_k(\theta) d\theta, \end{aligned} \tag{2.16}$$

$$\begin{aligned} U_{i,j} \approx u_0(x_i) + \frac{T(\tau_j+1)}{4} \Big\{ \sum_{n=0}^{N}\sum_{m=0}^{M} l_n(x_i)\Phi(x_n,\tau_m) + \sum_{n=1}^{N-1}\sum_{m=0}^{M} l_n''(x_i) U(x_n,\tau_m) \\ - p(x_i) \sum_{n=1}^{N-1}\sum_{m=0}^{M} l_n'(x_i) U(x_n,\tau_m) \Big\} \sum_{z=1}^{p-1} l_z(x_i) \sum_{k=0}^{p} F_m(\xi_{j,k}) \int_{-1}^{1} F_k(\theta) d\theta. \end{aligned} \tag{2.17}$$

Now we discuss an efficient way to find $d_k = \displaystyle\int_{-1}^{1} F_k(\theta) d\theta$ and express it in an explicit form. First we express $F_j(s)$ in terms of Chebyshev functions as in [27], [35]:

$$F_j(s) = \omega_j^C \sum_{p=0}^{M} (T_p(x_j)/\gamma_p) T_p(s), \tag{2.18}$$

where ω_j^C is the Chebyshev weight corresponding to Chebyshev points $\{x_j\}_{j=0}^M$. Let

$$\gamma_p = \sum_{i=0}^{M} T_p^2(x_i)\omega_i^C = \begin{cases} \pi, & p = 0, \\ \pi/2, & 1 \le p \le M, \end{cases} \tag{2.19}$$

where $\gamma_M = \pi/2$ if $\{x_i\}$ are the Chebyshev–Gauss or the Chebyshev–Gauss–Radau points. However, if we use $\{x_i\}$ as the Chebyshev–Gauss–Lobatto then $\gamma_M = \pi$. Now from equations (2.18) and (2.19), we can calculate d_k as follows

$$d_k = \omega_k^C \sum_{p=0}^{M} (T_p(x_k)/\gamma_p) \int_{-1}^{1} T_p(\theta)d\theta. \tag{2.20}$$

To compute $\int_{-1}^{1} T_p(\theta)d\theta$ we use the recurrence relation in equation (2.2) which yields

$$\int_{-1}^{1} T_p(\theta)d\theta = \begin{cases} 2/(1-p^2), & p \text{ is +ve even number}, \\ 0, & p \text{ is +ve odd number}. \end{cases} \tag{2.21}$$

Now rewrite equations (2.16) and (2.17) in the matrix form as

$$\begin{cases} \mathbf{\Phi}_{(\mathrm{N+1})(\mathrm{M+1})} = \mathbf{G}_{(\mathrm{N+1})(\mathrm{M+1})} + \mathbb{L}\mathbf{U}_{(\mathrm{N-1})(\mathrm{M+1})}, \\ \\ \mathbf{U}_{(\mathrm{N-1})(\mathrm{M+1})} = \mathbf{U}_{-1} + \mathbb{A}\mathbf{\Phi}_{(\mathrm{N+1})(\mathrm{M+1})} + (\mathbb{B}+\mathbb{C})\mathbf{U}_{(\mathrm{N-1})(\mathrm{M+1})}, \end{cases} \tag{2.22}$$

where $\mathbf{\Phi}, \mathbf{G}, \mathbf{U}$ and $\mathbf{U}_{-1}$ represent vectors defined as in the following:

$$\mathbf{\Phi}_{(\mathrm{N+1})(\mathrm{M+1})} = vec\left[\Phi_{i,j}\right], \quad \mathbf{G}_{(\mathrm{N+1})(\mathrm{M+1})} = vec[g_{i,j}], \quad 0 \le i \le N,$$
$$\mathbf{U}_{(\mathrm{N-1})(\mathrm{M+1})} = vec[U_{i,j}], \quad 1 \le i \le N, \quad 0 \le j \le M,$$
$$\mathbf{U}_{-1} = vec\begin{bmatrix} u_0(x_2) & u_0(x_3) & \cdots & u_0(x_{N-1}) \\ \vdots & \vdots & \vdots & \vdots \\ u_0(x_2) & u_0(x_3) & \cdots & u_0(x_{N-1}) \end{bmatrix},$$

in which the (vec) operator reshapes any matrix into a vector by placing columns of the matrix below each other from the first to the last. For the other matrices, each one can be described as block ones as the following:

- $\mathbb{A} = (A_j^i)$, where $\mathbb{A}$ is a matrix with a dimension of $(N+1)(N-1)\times(N+1)^2$ in which the first and last $(N+1)$ columns are zeros, and each block matrix (A_j^i) has a dimension of $(N+1)\times(N+1)$. For each $0 \le m \le M$, $0 \le j \le M$, $1 \le i \le N-1$, $1 \le n \le N-1$, the other entries of (A_j^i) can be given by the following formula

  $$(A_j^i)_{i,n+1} = \frac{T(\tau_j+1)}{4}\sum_{k=0}^{p} d_k F_m(\xi_{j,k}).$$

  In addition, the shape of the global matrix $\mathbb{A}$ will be

  $$\mathbb{A} = \begin{bmatrix} \mathbf{0} & A_j^1 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & A_j^{N-1} & \mathbf{0} \end{bmatrix}.$$

- $\mathbb{L} = (L_j^i)$, where $\mathbb{L}$ is a matrix with a dimension of $(N+1)^2\times(N+1)(N-1)$ in which the first and last $(N+1)$ rows are zeros, and each block matrix (L_j^i) has a dimension of $(N+1)\times(N+1)$. For each m, j, i, n, the other entries of (L_j^i) can be obtained by the following formula

  $$(L_j^i)_{i+1,n} = \frac{T(\tau_j+1)}{4}\sum_{k=0}^{p} d_k K(x_i, \tau_j - \rho_{j,k})F_m(\rho_{j,k})D_{i,n}^2,$$

- $\mathbb{B} = (B_j^i)$, where $\mathbb{B}$ is a matrix with a dimension of $(N+1)(N-1) \times (N+1)(N-1)$ in which each block matrix (B_j^i) has a dimension of $(N+1) \times (N+1)$. For each m, j, i, n, the entries of (B_j^i) can be calculated from the following relation

$$(B_j^i)_{i,n} = \frac{T(\tau_j + 1)}{4} \sum_{k=0}^{p} d_k F_m(\xi_{j,k}) D_{i,n}^2,$$

- Finally, $\mathbb{C} = (C_j^i)$ which is similar to the matrix $\mathbb{B}$. For each m, j, i, n, the entries of (C_j^i) can be found from the following relation

$$(C_j^i)_{i,n} = -\frac{T(\tau_j + 1)}{4} p(x_i) \sum_{k=0}^{p} d_k F_m(\xi_{j,k}) D_{i,n}.$$

To solve the coupled equations system in equation (2.22), we convert them to a linear algebraic system

$$\mathbf{A}\mathbf{U}_{(\mathrm{N}-1)(\mathrm{M}+1)} = \mathbf{F}, \tag{2.23}$$

where

$$\begin{aligned} \mathbf{A} &= (I - (\mathbb{B} + \mathbb{C}) - \mathbb{A}\mathbb{L}) \\ \mathbf{F} &= \mathbf{U}_{-1} + \mathbb{A}\mathbf{G}_{(\mathrm{N}+1)(\mathrm{M}+1)}. \end{aligned}$$

By solving the system of equations in (2.23), we obtain an approximation to $\mathbf{U}_{(\mathrm{N}-1)(\mathrm{M}+1)}$ and hence the approximation to the original problem for all $x \in (-1, 1)$ and $t \in [0, T]$ can be obtained by

$$u(x,t) \approx \sum_{n=1}^{N-1} \sum_{m=0}^{M} l_n(x) F_m(\frac{2}{T}t - 1) U(x_n, \tau_m).$$

3 Numerical Examples

In order to test the utility of the proposed new method, we apply the new scheme to the following two examples whose exact solutions are provided in each case. For both examples, we take $T = 1$, $p = N = M$, and consider $\{\theta_k\}_{k=0}^{N}$ as the Chebyshev-Gauss points as described in Section 2. To show the efficiency of the presented method the comparison with the exact solution introduced in the next tables. Also, to study the convergence behavior of the presented method [40], [41] the following laws for different values of N are applied.

- The $\|E\|_\infty$ error norm of the solution which is defined by

$$\|E\|_\infty = \|U(x,t) - u(x,t)\|_\infty = \max_{\substack{1 \le i \le N-1, \\ 1 \le j \le M}} |U_{i,j} - u(x_i, t_j)|,$$

- The $\|E\|_2$ error norm of the solution which is defined by

$$\|E\|_2 = \|U(x,t) - u(x,t)\|_2 = \left[\sum_{i=1}^{N-1} \sum_{m=1}^{M} \Big(U_{i,j} - u(x_i, t_j) \Big)^2 \right]^{1/2},$$

- The condition number $K_g(\mathbf{A})$ of the coefficient matrix $\mathbf{A}$ is given by

$$K_g(\mathbf{A}) = \|\mathbf{A}\|_g \|\mathbf{A}^{-1}\|_g, \qquad g = 2, \infty.$$

The condition number $K_g(\mathbf{A})$ has the following properties [42]:

1. The condition number shows that a small perturbation in initial data may produce a large amount of perturbation in the solution [43].
2. The condition number of an interpolation matrix gives information on the numerical stability of the interpolation process.
3. The condition number depends more on the separation distance than on the number N of centers [44].
4. The condition number grows with N with the number of interpolating points in order to produce an interpolation matrix that is well-conditioned enough to be inverted in finite precision arithmetic [45].
5. When the number of mesh points increases, the condition number of the matrix becomes very large and the matrix tends to be ill-conditioned [43].

In this study, all the mathematical computations were executed in double precision arithmetic using Matlab 7.9.0 (R2009b). To obtain a very accurate calculations, variable arithmetic precision (vpa) was used with digits being assigned to be (32). The program was executed on a second generation Intel Core i5–2410M, 2.3 Ghz Laptop. Finally, the CPU time for all operations for the solution of the entire problem is presented in Tables.

Example 3.1. *Consider the linear PPIDE (1.1)–(1.3) with the kernel $k(x,t) = xt$, with the variable function $p(x) = x^3 \sin(x)e^x/(1 + x^2)$, and the leading function*

$$\begin{aligned} f(x,t) = &\big(9x^4 - 8x^3 - 1 - p(x)(1 - 4x^4)\big)\sin(x + t) - 8x^4\big(t\cos(x) + \sin(x)\big) \\ &+\big(x^5 - x^4 - 12x^3 + 12x^2 - x + 1 - 4x^3p(x)\big)\cos(x + t) \\ &-\big(x^5 - 12x^3 - x\big)\big(t\sin(x) - \cos(x)\big), \end{aligned}$$

so that the exact solution will be $u(x,t) = (1 - x^4)\cos(x + t)$.

Table 1: $\|E\|_\infty$ error, $\|E\|_2$ error, condition number of $g = \infty,\ g = 2$ and CPU time with different values of N for Example 3.1.

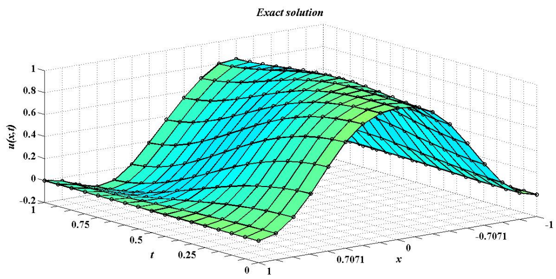
N	$\|E\|_\infty$	$K_\infty(\mathbf{A})$	$\|E\|_2$	$K_2(\mathbf{A})$	CPU(s)
6	1.913E-03	3.128E+02	4.659E-03	6.159E+01	5.775
8	1.527E-05	1.143E+03	4.373E-05	1.820E+02	11.057
10	7.030E-08	3.057E+03	2.407E-07	4.358E+02	19.473
12	2.013E-10	6.887E+03	8.215E-10	8.970E+02	31.442
14	3.883E-13	1.346E+04	1.825E-12	1.657E+03	53.030
16	1.346E-14	2.429E+04	5.186E-14	2.825E+03	125.18

Example 3.2. *Consider the linear PPIDE (1.1)-(1.3) with the kernel $k(x,t) = \sin(xt)$, with the variable function $p(x) = x^3\cos(x)e^x/(1 + x^2)$ and the leading function*

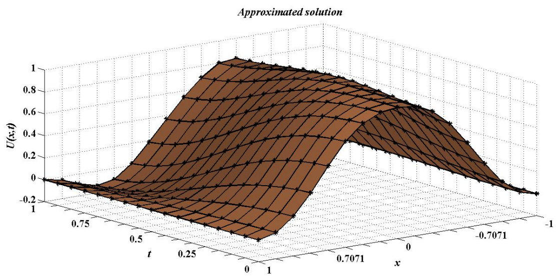
$$\begin{aligned} f(x,t) = &\big(8x^3 + 12x^2 + p(x)(1 - 4x^3 - x^4)\big)e^{(x+t)} \\ &+\Big(\frac{x^4 + 12x^2 + 8x^3 - 1}{1 + x^2}\Big)\big(xe^t - \sin(xt) - x\cos(xt)\big)e^x, \end{aligned}$$

so that the exact solution will be $u(x,t) = (1 - x^4)e^{(x+t)}$.

In **Figure 1**, the exact solution, numerical solution are plotted with $\|E\|_\infty$ and the nonuniform set of collocation points (x_i, t_j) at $N = 16$.



(a) Exact solution



(b) Approximated solution

Figure 1: Exact and Approximated solutions for $p(x)$ with $x \in [-1, 1]$ and $t \in [0, T]$ at $N = 16$, see Example 3.1.

Table 2: $\|E\|_\infty$ error, $\|E\|_2$ error, condition number of $g = \infty$, $g = 2$ and CPU time with different values of N for Example 3.2.

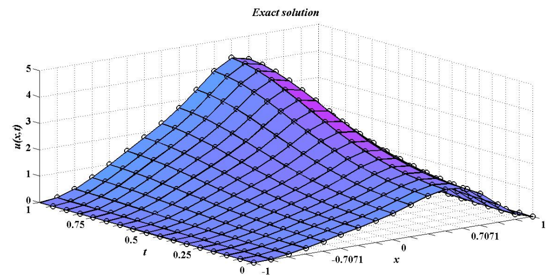
N	$\|E\|_\infty$	$K_\infty(\mathbf{A})$	$\|E\|_2$	$K_2(\mathbf{A})$	CPU(s)
6	7.414E-03	3.094E+02	1.677E-02	6.154E+01	8.928
8	5.673E-05	1.310E+03	1.563E-04	1.817E+02	12.828
10	2.597E-07	3.035E+03	9.047E-07	4.350E+02	23.067
12	7.218E-10	6.823E+03	2.975E-09	8.952E+02	34.251
14	1.367E-12	1.336E+04	6.598E-12	1.653E+03	53.734
16	7.105E-14	2.409E+04	1.858E-13	2.818E+03	115.770

Example 3.3. *Consider the linear PPIDE (1.1)-(1.3) with the kernel* $k(x,t) = e^{-x^2 t}$*, with the variable function* $p(x) = \cos(x)/(1 + x^2)$*, and the function*

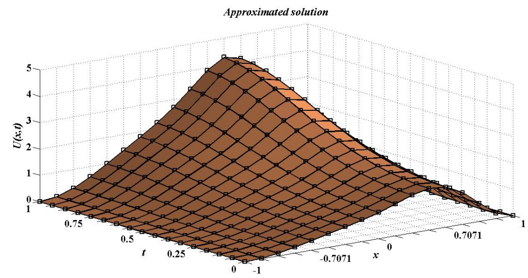
$$f(x,t) = \big(12x^5 + 30x^4 + p(x)(1 - 6x^5 - x^6)\big)e^{(x+t)}$$
$$-\Big(\frac{1 - 30x^4 - 12x^5 - x^6}{1 + x^2}\Big)\big(e^{x+t} - e^{(x^2+t)}\big),$$

so that the exact solution will be $u(x,t) = (1 - x^6)e^{x+t}$*.*

In **Figure 2**, the exact solution, numerical solution are plotted with $\|E\|_\infty$ and the nonuniform set of collocation points (x_i, t_j) at $N = 16$.



(c) Exact solution



(d) Approximated solution

Figure 2: Exact and Approximated solutions for $p(x)$ with $x \in [-1, 1]$ and $t \in [0, T]$ at $N = 16$, see Example 3.2.

Table 3: $\|E\|_\infty$ error, $\|E\|_2$ error, condition number of $g = \infty,\ g = 2$ and CPU time with different values of N for Example 3.3.

N	$\|E\|_\infty$	$K_\infty(\mathbf{A})$	$\|E\|_2$	$K_2(\mathbf{A})$	CPU(s)
8	1.271E-03	1.507E+03	3.452E-03	2.436E+02	8.910
10	1.412E-05	3.999E+03	3.858E-05	5.698E+02	16.344
12	5.329E-08	8.904E+03	2.186E-07	1.156E+03	33.014
14	1.529E-10	1.726E+04	7.384E-10	2.115E+03	54.946
16	3.341E-13	3.099E+04	1.709E-12	3.581E+03	100.49
18	1.221E-13	5.155E+04	4.438E-13	5.705E+03	288.43

Tables "**1** $-$ **3**", show the absolute (*Error*) between the exact and numerical solutions, $\|E\|_\infty$-error, $K_\infty(\mathbf{A})$, $\|E\|_2$-error, $K_2(\mathbf{A})$ and CPU-time in some points of the interval $(-1, 1)$ and $t = [0, T]$ for $6 \le N \le 18$. These tables indicates that as N increases, the Error decreases more rapidly (exponentially). From Tables "1– 3", it can be observed that the accuracy increases with the increase of number of collocation points.

4 Conclusion

In this work, a new modified numerical method based on developing the Chebyshev spectral–collocation method. The method is based on converting the problem equation into two coupled Volterra integral equations of the second kind. The use of the Chebyshev spectral–collocation method lead to solving a linear algebraic system, then finding the approximate solution for parabolic partial integro–differential equations. The major progress of using the Chebyshev polynomials basis instead of using Legendre ones is that its quadrature points are known with explicit and simple expressions as well as the corresponding weights. Moreover, in this work the integration was approximated by replacing the integrand by its interpolating polynomials instead of using Gauss quadrature approximation and this increases the accuracy of the suggested method. The examples given in this work have demonstrated the potential of the newly presented numerical method in solving parabolic partial integro–differential and similar equations even with using a small number of collocation points. The stability of the resulting system was proved by helping of the condition number $K_g(\mathbf{A})$.

Acknowledgment

The authors are very grateful to both reviewers for carefully reading the paper and for their comments and suggestions which helped to improve the paper.

Competing Interests

The authors declare that no competing interests exist.

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