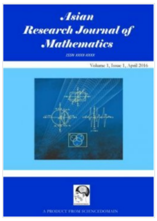

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Optimal System of Subalgebras for the Time Fractional Generalized Burgers Equation

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Authors' contributions

This work was carried out in collaboration between all authors. Author HAZ designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors HAZ and SST managed the analyses of the study. All authors read and approved the final manuscript.

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Review Article

Abstract

This article is devoted to use the basic Lie symmetry method to obtain the admitted Lie group of the reduction of Time fractional generalized Burgers equation. One-dimensional and two-dimensional optimal system are determined for symmetry algebras obtained through classification of their subalgebras.Some invariant solutions are also found.

Keywords: Fractional differential equations; time fractional generalized Burgers equation; optimal system of subalgebras.

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1 Introduction

Fractional differential equations useful in various fields such as viscoelasticity, fluid mechanics, physics, biology, engineering and other areas of science [1]. In reality, a physical phenomenon may depend not only on the time instant but also on the previous time history, which can be successfully modeled by using the theory of derivatives and integrals of fractional order [2],[3] and [4].

Symmetry is a basic property of nature and its phenomena. Therefore, the equations that are able to describe chemical ,physical or biological processes should have symmetry properties. The investigated processes can be described by using ordinary or partial differential equations. Symmetry properties of these equations are usually investigated by using group analysis methods. The fractional differential equation (FDE) is another type of equation which used in mathematical modeling of physical processes during the last decade.

Since Lie introduced the notion of continuous transformation group, now known as Lie group, the theory of Lie groups and Lie algebras have been evolved into one of the most explosive development of mathematics and physics through out the past century [5].

Lie groups and hence their infinitesimal generators can be extended to act on the space of independent variables, state variables (dependent variables) and derivatives of the state variables up to any finite order. Lie symmetries were introduced by Lie in order to solve ordinary differential equations. Another application of symmetry methods is to reduce systems of differential equations, finding equivalent systems of differential equations of simpler form.

This paper is dedicated to use the basic Lie symmetry method for finding the admitted Lie group of the reduction of Time fractional generalized Burgers equation [6], [7]:

$$u_t^{\alpha} = u_{xx}^{\beta} + Au^c u_x^{\beta}, \qquad 0 < \alpha, \beta \leq 1, c0. \tag{1.1}$$

The space-fractional Burgers equation describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer. The same form can be found in other systems such as shallow-water waves and waves in bubbly liquids. For more details about the associated applications of the Time fractional generalized Burgers equation see [8].

In the case $c = 1$ and $A = 1$; Eq.(1.1) will be in the following form:

$$u_t^{\alpha} = u_{xx}^{\beta} + u\ u_x^{\beta}, \qquad 0 < \alpha, \beta \leq 1, \tag{1.2}$$

where α and β are areal constants. Here u is function of independent variables x and t; $D^{\alpha}u$ is a fractional derivative of u with respect to x, which can be of Riemann–Liouville type [9], [10]

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}} = \begin{cases} \frac{\partial^m u}{\partial x^m}, & \alpha = m \in \mathbb{N}; \\ \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial x^m} \int_0^t (t-\tau)\, m^{-\alpha-1} u(\tau, x)\, d\tau, & m-1 < \alpha < m,\ m \in \mathbb{N}. \end{cases} \tag{1.3}$$

2 Preliminaries

Guy Jumarie [11] proposed some modifications in Riemann–Liouville fractional derivative and derived fractional. The new modified fractional derivative has some features similar to the classical derivative. The definition and properties of modified fractional derivatives are defined below [12]

$$D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (f - f(0))\,(x-\xi)^{-\alpha-1}\, d\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (f(\xi) - f(0))\,(x-\xi)^{-\alpha}\, d\xi, & 0 < \alpha \leq 1, \\ \left(f^{(n-1)}(x)\right)^{\alpha-n+1}, & n-1 < \alpha \leq n,\ n \geq 2 \end{cases} \tag{2.1}$$

The modified Riemann–Liouville fractional derivative endures some interesting properties:

$$D_x^\alpha x^\mu = \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}, \quad \mu 0, \tag{2.2}$$

$$D_x^\alpha (f(x)\, g(x)) = g(x)\, D_x^\alpha f(x) + f(x)\, D_x^\alpha g(x), \tag{2.3}$$

$$D_x^\alpha f[g(x)] = \frac{df[g(x)]}{dg(x)} D_x^\alpha g(x). \tag{2.4}$$

3 Time Fractional Generalized Burgers Equation

Consider Time fractional generalized Burgers equation will be in the following form:

$$u_t^\alpha = u_{xx}^\beta + u\, u_x^\beta, \qquad 0 < \alpha, \beta \leq 1, \tag{3.1}$$

We introduce the transformations [12]

$$X = \frac{p x^\beta}{\Gamma(1+\beta)}, \quad T = \frac{q t^\alpha}{\Gamma(1+\alpha)}, \quad W(T, X) = u(t, x), \quad p, q \neq 0. \tag{3.2}$$

Equation (3.1) with the help of (2.2), (2.4) and (3.2) transforms to second-order partial differential equation

$$q W_T - p^2 W_{XX} - W W_X = 0. \tag{3.3}$$

Using mathematica, the Lie symmetries of Eq.(3.3) are

$$\begin{aligned}
\chi_1 &= \frac{\partial}{\partial T}, \\
\chi_2 &= \frac{\partial}{\partial X}, \\
\chi_3 &= 2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X} - W\frac{\partial}{\partial W}, \\
\chi_4 &= T\frac{\partial}{\partial X} - q\frac{\partial}{\partial W}, \\
\chi_5 &= TX\frac{\partial}{\partial X} + T^2\frac{\partial}{\partial T} - (TW + qX)\frac{\partial}{\partial W}
\end{aligned} \tag{3.4}$$

4 Optimal System of Subalgebras

There are two basic issues concerning optimal systems, the determination of all subalgebras and the classification of group invariant solutions [13]. One of the main applications of Lie theory of symmetry groups for differential equations is the construction of group invariant solutions [14], [15]. Given any subgroup of the symmetry group, one can write down the equation for the invariant solution with respect to this subgroup. This reduced equation is of fewer variables and is easier to solve generally [16].

The problem is to construct subalgebras of the algebra $_5$, which can be a source of invariant solutions of Eq.(3.3). The classification of subalgebras can be done relatively easy for small dimensions. The optimal system of subalgebras of the Lie algebra spanned by the generators $\chi_1, \chi_2, \chi_3, \chi_4$ and χ_5

are constructed here.

Thus, corresponding commutator table of $\{\chi_i; (i = 1, 2, 3, 4, 5)\}$ can be constructed (in the following table) [17], [18]:

Table 1. Commutator$[X_i, X_j] = X_iX_j - X_jX_i$.

	χ_1	χ_2	χ_3	χ_4	χ_5
χ_1	0	0	$-2\chi_1$	$-\chi_2$	$-\chi_3$
χ_2	0	0	$-\chi_2$	0	$-\chi_4$
χ_3	$2\chi_1$	χ_2	0	$-\chi_4$	$-2\chi_5$
χ_4	χ_2	0	χ_4	0	0
χ_5	χ_3	χ_4	$2\chi_5$	0	0

It is easy to check that $\{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$ are closed under the Lie bracket. Thus, a basis for the Lie algebra is $\{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$, which is a 5−dimensional Lie group algebra.

4.1 Decomposition of the Algebra $_5$

Before constructing an optimal system, let us study the algebraic structure of the algebra $_5$. The algebra $_5$ is decomposed as $I \oplus _3$, where $I = \{\chi_2, \chi_4\}$ is an ideal and $_3 = \{\chi_1, \chi_3, \chi_5\}$ is a subalgebra. According to the algorithm for constructing an optimal system of the algebra $_5$, we use the two-step algorithm. First, an optimal system of subalgebras of the algebra $_3$ is obtained. The next step is to glue the subalgebras from the optimal system of subalgebras of the algebra $_3$ and the ideal I together.

4.2 Classification of the Algebra $_3$

The table of commutators of the algebra $_3 = \{\chi_1, \chi_3, \chi_5\}$ is

Table 2. Commutator of $_3$

	χ_1	χ_3	χ_5
χ_1	0	$-2\chi_1$	$-\chi_3$
χ_3	$2\chi_1$	0	$-2\chi_5$
χ_5	χ_3	$2\chi_5$	0

Let a subalgebra $_r$ of dimension $r \leq 3$ be formed by the operators

$$Y_i = a_{i1}\chi_1 + a_{i3}\chi_3 + a_{i5}\chi_5, \quad i = 1, \ldots, r \tag{4.1}$$

where $a_{ij}, (i = 1, \ldots, r; j = 1, 2, 3)$ are arbitrary constants. For the classification of $_3$, we need to study two steps [19]:

1) All coefficients a_{i3} are zero, $a_{i3} = 0\,(i = 1, 2, 3)$, it means that we will construct an optimal system of the subalgebra $_3 = \{\chi_1, \chi_3, \chi_5\}$.

2) At least one of the coefficients of a_{i3} is not equal to zero.

We will denote the generators χ_i by i.

4.2.1 One-Dimensional Subalgebras of the Algebra $_3$

Let $Y = x_1 1 + x_3 3 + x_5 5$ which forms a one-dimensional subalgebra of the algebra $_3$. The process of simplification of the coefficients of the operator Y is separated into the following cases.

Case 4.1. *Assume that $x_5 \neq 0$. Then one can divide Y by x_5. We have two subcases*
Case 1.1 When $x_1 \neq 0$ and $x_3 = 0$, the operator Y is transformed to $\epsilon 1 + 5$ where $\epsilon = \pm 1$.
Case 1.2 When $x_3 \neq 0$ and $x_1 = 0$, the operator Y is transformed to $\epsilon 3 + 5$ where $\epsilon = \pm 1$.

Case 4.2. *Assume that $x_5 = 0$. We have three subcases*
Case 2.1 When $x_1 \neq 0$ and $x_3 = 0$, the operator Y is transformed to 1.
Case 2.2 When $x_3 \neq 0$ and $x_1 = 0$, the operator Y is transformed to 3.
Case 2.3 When $x_1 \neq 0$ and $x_3 \neq 0$, the operator Y is transformed to $\epsilon 1 + 3$.

Case 4.3. *Assume that $x_3 \neq 0$. Then one can divide Y by x_3. We have two subcases*
Case 3.1 When $x_1 \neq 0$ and $x_5 = 0$, the operator Y is transformed to $\epsilon 1 + 3$ where $\epsilon = \pm 1$.
Case 3.2 When $x_5 \neq 0$ and $x_1 = 0$, the operator Y is transformed to $\epsilon 3 + 5$ where $\epsilon = \pm 1$.

Case 4.4. *Assume that $x_3 = 0$. We have three subcases*
Case 4.1 When $x_1 \neq 0$ and $x_5 = 0$, the operator Y is transformed to 1.
Case 4.2 When $x_5 \neq 0$ and $x_1 = 0$, the operator Y is transformed to 5.
Case 4.3 When $x_1 \neq 0$ and $x_5 \neq 0$, the operator Y is transformed to $\epsilon 1 + 5$.

Case 4.5. *Assume that $x_1 \neq 0$. Then one can divide Y by x_5. We have two subcases*
Case 5.1 When $x_5 \neq 0$ and $x_3 = 0$, the operator Y is transformed to $\epsilon 1 + 5$ where $\epsilon = \pm 1$.
Case 5.2 When $x_3 \neq 0$ and $x_5 = 0$, the operator Y is transformed to $\epsilon 3 + 5$ where $\epsilon = \pm 1$.

Case 4.6. *Assume that $x_1 = 0$. We have three subcases*
Case 6.1 When $x_5 \neq 0$ and $x_3 = 0$, the operator Y is transformed to 5.
Case 6.2 When $x_3 \neq 0$ and $x_5 = 0$, the operator Y is transformed to 3.
Case 6.3 When $x_5 \neq 0$ and $x_3 \neq 0$, the operator Y is transformed to $\epsilon 3 + 5$.

4.2.2 Two-Dimensional Subalgebras of the Algebra $_3$

Let a subalgebra be formed by the operators

$$Y_i = a_{i1}\chi_1 + a_{i3}\chi_3 + a_{i5}\chi_5, \quad i = 1, 2 \tag{4.2}$$

where $a_{ij}, (i = 1, 2; j = 1, 2, 3)$ are arbitrary constants. The rank of the matrix $\begin{pmatrix} a_{11} & a_{13} & a_{15} \\ a_{21} & a_{23} & a_{25} \end{pmatrix}$ is equal to two.

Case 4.7. *Assume that $a_{15} \neq 0$, $a_{25} = 0$ and $a_{21}^2 + a_{23}^2 \neq 0$. Then one can divide Y_1 by x_5. We have two subcases*

Case 7.1 If $a_{13} = 0$, then the subalgebra condition gives

$$[a_{11}1 + 5, a_{21}1 + a_{23}3] = \alpha (a_{11}1 + 5) + \beta (a_{21}1 + a_{23}3) \tag{4.3}$$

where α and β are an arbitrary constants. Calculating the left hand side and comparing the coefficients on the right hand side with coefficients on the left hand side, one obtain

$$-2a_{11}a_{23}1 + a_{21}3 + 2a_{23}5 = (\alpha a_{11} + \beta a_{21})\, 1 + \beta a_{23}3 + \alpha 5. \tag{4.4}$$

Therefore

$$-2a_{11}a_{23} = \alpha a_{11} + \beta a_{21}, a_{21} = \beta a_{23} \text{ and } 2a_{23} = \alpha. \tag{4.5}$$

We have two subcases

Case 7.1.1 If $a_{23} = 0$, then $a_{21} = 0$ which is contraction with $a_{21}^2 + a_{23}^2 \neq 0$.
Case 7.1.2 If $a_{23} \neq 0$ and $a_{23} = 1$, then $\alpha = 2, \beta = a_{21}$ and $4a_{11} = -\ a_{21}^2$. We find that
(a) If $a_{21} = 0$, then the operators Y_1 and Y_2 are transformed to $Y_1 = 5, Y_2 = 3$.

(b) If $a_{21} \neq 0$, then the operators Y_1 and Y_2 are transformed to $Y_1 = 5 - \gamma 1, Y_2 = 3 + 1$ where $\gamma = \frac{1}{4}$.

Case 7.2 If $a_{13} \neq 0$, We have two cases

Case 7.2.1 If $a_{23} = 0$, then by exchanging Y_1 and Y_2, this becomes the previous case.

Case 7.2.2 If $a_{23} \neq 0$, then the operators are $Y_1 = a_{11}1 + a_{13}3$ and $Y_2 = a_{21}1 + a_{23}3$ because the rank of the matrix $\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}$ is equal to two. By taking linear combinations of the operators Y_1 and Y_2 they can be transformed to $Y_1 = 1$, $Y_2 = 3$.

4.2.3 Three-Dimensional Subalgebras of the Algebra $_3$

Let a subalgebra be formed by these operators

$$Y_i = a_{i1}\chi_1 + a_{i3}\chi_3 + a_{i5}\chi_5, \quad i = 1, 2, 3 \tag{4.6}$$

where $a_{ij}, (i = 1, 2, 3; j = 1, 2, 3)$ are arbitrary constants. The rank of the matrix $\begin{pmatrix} a_{11} & a_{13} & a_{15} \\ a_{21} & a_{23} & a_{25} \\ a_{31} & a_{33} & a_{35} \end{pmatrix}$ is equal to three. Therefore, the basis of this subalgebra can be taken as $Y_1 = 1$, $Y_2 = 3$ and $Y_3 = 5$.

4.2.4 Optimal System of Subalgebras of the Algebra $_3$

The result of classifying the algebra $_3 = \{\chi_1, \chi_3, \chi_5\}$ is given in the following table:

Table 3. Classifying of the algebra $_3$

Dimension		
1	2	3
3	$1, 3$	$1, 3, 5$
5	$1, 5$	
$\epsilon 1 + 3$	$3, 5$	
$\epsilon 1 + 5$	$5 - \gamma 1, 3 + 1$	
$\epsilon 3 + 5$	$3 + 5, 1 + \gamma 3$	
	$1 - \gamma 5, 3 + 5$	

5 Optimal System of Subalgebras of the Algebra $_5$

After constructing an optimal system of subalgebras of the algebra $_3$, the next step is the construction of an optimal system of subalgebras of the algebra $_5 = \{1, 2, 3, 4, 5\}$, by gluing subalgebras from the optimal system of subalgebras of the algebra $_3$ and the ideal $I = \{\chi_2, \chi_4\}$ together [19]. This process consists of the following steps. In the first step, the vectors

$$\begin{aligned} Y_i &= \sum_{j=\{2,4\}} a_{ij}\chi_j + \sum_{j=\{1,3,5\}} b_{ij}\chi_j, \quad (i=1,2,\ldots,k), \\ Y_{i+k} &= \sum_{j=\{2,4\}} c_{ij}\chi_j, \quad (i=1,2,\ldots,s), \end{aligned} \tag{5.1}$$

are composed. Here the vectors $\sum_{j=\{1,3,5\}} b_{ij}\chi_j$ are basis elements from one of the k-dimensional subalgebras $_k$ of the optimal system of the algebra $_3$.In matrix form, this step can be explained by the construction of the matrix

2, 4	1, 3, 5
A	C
B	0

where A, B and C consist of the coefficients $a_{ij}, b_{i\alpha}$ and $c_{\beta j}, (i=1,2,\ldots,k; j=2,4; \alpha=1,3,5; \beta=1,2,\ldots,s)$.

In the following step, the matrix A is arbitrary. The rank of the matrix $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}$ is equal to $k+s$ and this is the dimension of the subalgebra of the algebra $_5$. The matrix A can be simplified with the help of the matrix C. The matrix C has to take all possible values of the given rank s and is chosen by taking linear combinations of it columns. The next step is the process of checking the subalgebra conditions.

Let us give an example for constructing subalgebras, using the subalgebra $\{\epsilon 1 + 5\}$. In this case, the matrix C is 1×2 a matrix:

2	4	1	3	5
a_{12}	a_{14}	ϵ	0	5
c_{22}	c_{24}	0	0	0

The subalgebra conditions give

$$\begin{aligned} [\epsilon 1 + 5 + a_{12}2 + a_{14}4, c_{22}2 + c_{24}4] &= c_{22}4 - \epsilon c_{24}2 \\ &= \alpha\left(\epsilon 1 + 5 + a_{12}2 + a_{14}4\right) + \beta c_{22}2 + c_{24}4 \end{aligned} \tag{5.2}$$

where α and β are an arbitrary constants. By comparing the coefficients, we find that $\alpha = 0$ and have the following two cases:

Case 5.1. *If $c_{22} = c_{24} = 0$, we get this subalgebras $\{\epsilon 1 + 2 + 5\}$, $\{\epsilon 1 + 4 + 5\}$ and $\{\epsilon 1 + 2 + 4 + 5\}$.*

Case 5.2. *If $c_{22} \neq 0, c_{24} \neq 0$ and $c_{22} = c_{24} = 1$, we get this subalgebras $\{\epsilon 1 + 5, 2 + 4\}$, $\{\epsilon 1 + 2 + 5, 2 + 4\}$, $\{\epsilon 1 + 4 + 5, 2 + 4\}$ and $\{\epsilon 1 + 2 + 4 + 5, 2 + 4\}$ where $\beta = \pm\sqrt{\epsilon} i$.*

The list of one-dimensional and two-dimensional subalgebras of the optimal system of the algebra $_5$ is presented in the following table:

Table 4. Classifying of the algebra $_5$

Dimension	
1	2
$1+2$	$3+4, 2$
$1+4$	$3+2, 2$
$1+2+4$	$3+4, 4$
$2+5$	$3+2, 4$
$4+5$	$\epsilon 1+3+4, \delta 2+4$
$2+4+5$	$\epsilon 1+2+3, \delta 2+4$
$2+3$	$\epsilon 1+3, \delta 2+4$
$3+4$	$\epsilon 1+2+3+4, \delta 2+4$
$2+3+4$	$\epsilon 1+2+5, 2+4$
$\epsilon 1+2+3$	$\epsilon 1+4+5, 2+4$
$\epsilon 1+3+4$	$\epsilon 1+5, 2+4$
$\epsilon 1+2+3+4$	$\epsilon 1+2+4+5, 2+4$
$\epsilon 1+2+5$	$\epsilon 3+2+5, 2+\mu 4$
$\epsilon 1+4+5$	$\epsilon 3+4+5, 2+\mu 4$
$\epsilon 1+2+4+5$	$\epsilon 3+5, 2+\mu 4$
$\epsilon 3+2+5$	$\epsilon 3+2+4+5, 2+\mu 4$
$\epsilon 3+4+5$	
$\epsilon 3+2+4+5$	

where $\delta = \frac{\epsilon}{2}$ and $\mu = \frac{1}{2\epsilon}$.

6 Invariant Solutions of Time fractional generalized Burgers Equation (3.1)

In this section, invariant solutions of Equation (3.1) are presented. Analysis of invariant solutions is presented in details for the following two examples.

6.1 For the subalgebra $\{1\}$:

The basis of this subalgebra is

$$\chi_1 = \frac{\partial}{\partial T}. \tag{6.1}$$

The similarity variable is:

$$Z = X, \tag{6.2}$$

and the similarity function is:

$$W = F(Z). \tag{6.3}$$

Substituting (6.3) into Eq.(3.3), we finally obtain an ordinary differential equation for $F(Z)$ take the form:

$$p^2 F''(Z) + F'(Z) F(Z) = 0. \tag{6.4}$$

Solving the above ordinary differential Eq.(6.4), we have the following solution:

$$F(Z) = \sqrt{2C_1} p Tanh\left[\frac{\sqrt{2C_1}Z + \sqrt{2C_1}C_2}{2p}\right], \tag{6.5}$$

where $C_1$and C_2 are arbitrary constants. Substituting from Eq.(6.5) into Eq.(6.3), then in Eq.(3.2) to obtain the solution for Eq.(3.1) in the following form:

$$u(x,t) = \sqrt{2C_1}pTanh\left[\frac{\sqrt{2C_1}\frac{px^\beta}{\Gamma(1+\beta)} + \sqrt{2C_1}C_2}{2p}\right], \tag{6.6}$$

where $C_1$and C_2 are arbitrary constants.

6.2 For the subalgebra $\{1,3\}$:

The basis of this subalgebra is

$$\begin{aligned}\chi_1 &= \frac{\partial}{\partial T},\\ \chi_3 &= 2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X} - W\frac{\partial}{\partial W}.\end{aligned} \tag{6.7}$$

Let a function $f = f(T,X,W)$ be an invariant of the generator χ_3. This means that

$$2Tf_T + Xf_X - Wf_W = 0, \tag{6.8}$$

The characteristic system of the last equation is given as:

$$\frac{dT}{2T} = \frac{dX}{X} = -\frac{dW}{W}, \tag{6.9}$$

the similarity variable is:

$$\hat{X} = TX^{-2}, \tag{6.10}$$

and the similarity function is:

$$\hat{W} = XW. \tag{6.11}$$

After substituting it into the equation $\chi_1 f = 0$, where $f = f\left(T,\hat{X},\hat{W}\right)$,the similarity variable is:

$$\hat{X} = Z, \tag{6.12}$$

and the similarity function is:

$$\hat{W} = \phi(Z). \tag{6.13}$$

Hence, a representation of the invariant solution is

$$W = X^{-1}\phi(Z), \tag{6.14}$$

where $\phi(Z)$ is an arbitrary function and $Z = TX^{-2}$.

Substituting (6.14) into Eq.(3.3), we finally obtain an ordinary differential equation for $\phi(Z)$ take the form:

$$-4p^2Z^2\phi''(Z) + 2Z\phi'(Z)\phi(Z) - \left(10p^2Z - q\right)\phi'(Z) + \phi(Z)^2 - 2p^2\phi(Z) = 0. \tag{6.15}$$

Solving the above ordinary differential Eq.(6.15), we have the following solution:

$$\begin{aligned}\phi(Z) \quad = \quad & -(4p^2 Z(-\frac{\sqrt{q}\left(\frac{1}{Z}\right)^{\frac{3}{2}}}{4p} Hypergeometric1F_1\left[\frac{q-2C_1}{2q},\frac{3}{2},-\frac{q}{4p^2 Z}\right] \\ & -\frac{C_1 C_2}{2p^2 Z^2} Hypergeometric1F_1\left[1-\frac{C_1}{q},\frac{3}{2},-\frac{q}{4p^2 Z}\right] \\ & +\frac{q^{\frac{3}{2}}\left(\frac{1}{Z}\right)^{\frac{5}{2}}\left(\frac{q-2C_1}{2q}\right)}{12p^3} Hypergeometric1F_1\left[\frac{3q-2C_1}{2q},\frac{5}{2},-\frac{q}{4p^2 Z}\right])) \\ & /(C_2 Hypergeometric1F_1\left[-\frac{C_1}{q},\frac{1}{2},-\frac{q}{4p^2 Z}\right] + \\ & \frac{\sqrt{\frac{q}{Z}}}{2p} Hypergeometric1F_1\left[\frac{q-2C_1}{2q},\frac{3}{2},-\frac{q}{4p^2 Z}\right]), \end{aligned} \tag{6.16}$$

where $C_1$and C_2 are arbitrary constants. Substituting from Eq.(6.16) into Eq.(6.14), then in Eq.(3.2) to obtain the solution for Eq.(3.1) in the following form:

$$\begin{aligned} u(x,t) \quad = \quad & -\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)\times(4p^2\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)^2(-\frac{\sqrt{q}}{4p}\frac{\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)^3}{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^{\frac{3}{2}}} \\ & \times Hypergeometric1F_1\left[\frac{q-2C_1}{2q},\frac{3}{2},-\frac{q\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)^2}{4p^2\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)}\right]-\frac{C_1C_2\left(\frac{px^\beta}{\Gamma(1+\beta)}\right)}{2p^2\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2} \\ & \times Hypergeometric1F_1\left[\frac{q-C_1}{q},\frac{3}{2},-\frac{q\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)^2}{4p^2\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)}\right]+\frac{q^{\frac{3}{2}}}{12p^3}\frac{\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)^5}{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^{\frac{5}{2}}} \\ & \left(\frac{q-2C_1}{2q}\right) Hypergeometric1F_1\left[\frac{3q-2C_1}{2q},\frac{5}{2},-\frac{q\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)^2}{4p^2\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)}\right])) \\ & /(C_2 Hypergeometric1F_1\left[-\frac{C_1}{q},\frac{1}{2},-\frac{q\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)^2}{4p^2\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)}\right] \\ & +\frac{\sqrt{\frac{q\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)^2}{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)}} Hypergeometric1F_1\left[\frac{q-2C_1}{2q},\frac{3}{2},-\frac{q\left(\frac{\Gamma(1+\beta)}{px^\beta}\right)^2}{4p^2\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)}\right]}{2p}), \end{aligned} \tag{6.17}$$

where $C_1$and C_2 are arbitrary constants.

The list of the invariant solution of subalgebras of the Algebra $_5$ which give the solution for the time fractional generalized Burgers Equation (3.1) is presented in the following table:

Table 5. Invariant solution of subalgebras 3, 5 and $\epsilon 1+3$.

No	*subalgebra*	*invariant*	*solution for Eq.(3.1)*
1	3	$Z=XT^{-\frac{1}{2}}$ $W=T^{-\frac{1}{2}}F(Z)$	$u(x,t)=\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}(2p^{2}(C_{2}(-\frac{q\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)}{2p^{2}}$ $\times HermiteH\left[\frac{-q+C_{1}}{q},\frac{\sqrt{q}}{2p}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)\right])+\frac{(-q+C_{1})}{p\sqrt{q}}$ $\times HermiteH\left[-1+\frac{-q+C_{1}}{q},\frac{\sqrt{q}}{2p}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)\right]$ $-\frac{q\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)}{2p^{2}}$ $\times Hypergeometric1F_{1}\left[-\frac{-q+C_{1}}{2q},\frac{1}{2},\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)^{2}}{4p^{2}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]$ $-\frac{(-q+C_{1})}{2p^{2}}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)$ $\times Hypergeometric1F_{1}\left[1-\frac{-q+C_{1}}{2q},\frac{3}{2},\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)^{2}}{4p^{2}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]))$ $/(C_{2}HermiteH\left[\frac{-q+C_{1}}{q},\frac{\sqrt{q}}{2p}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)\right]$ $+Hypergeometric1F_{1}\left[-\frac{-q+C_{1}}{2q},\frac{1}{2},\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)^{2}}{4p^{2}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]),$
2	5	$Z=\frac{X}{T}$ $W=\frac{F(Z)}{T}-qZ$	$u(x,t)=-\frac{\Gamma(1+\alpha)}{t^{\alpha}}^{-1}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)+\frac{\sqrt{2C_{1}}p\Gamma(1+\alpha)}{qt^{\alpha}}$ $\times Tanh\left[\frac{\sqrt{2C_{1}}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-1}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)+\sqrt{2C_{1}}C_{2}}{2p}\right],$
3	$\epsilon 1+3$	$Z=\frac{X}{\sqrt{2T+\epsilon}}$ $W=\frac{F(Z)}{\sqrt{2T+\epsilon}}$	$u(x,t)=\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)^{-\frac{1}{2}}(2p^{2}(C_{2}(-\frac{q}{p^{2}}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)$ $\times HermiteH\left[\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}}{\sqrt{2}p}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)\right]$ $+\frac{(-2q+C_{1})}{p\sqrt{2q}}$ $\times HermiteH\left[-1+\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}}{\sqrt{2}p}\left(2\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)\right])$ $-\frac{q}{p^{2}}\left(2\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)$ $\times Hypergeometric1F_{1}\left[-\frac{-2q+C_{1}}{4q},\frac{1}{2},\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)^{2}}{2p^{2}\left(2\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)}\right]$ $-\left(\frac{-2q+C_{1}}{2p^{2}}\right)\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)^{-\frac{1}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)\times$ $Hypergeometric1F_{1}\left[\frac{6q-C_{1}}{4q},\frac{3}{2},\frac{qx^{2\beta}}{2\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)(\Gamma(1+\beta))^{2}}\right]))$ $/(C_{2}HermiteH\left[\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}x^{\beta}}{\sqrt{2\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)}\Gamma(1+\beta)}\right]+$ $Hypergeometric1F_{1}\left[\frac{2q-C_{1}}{4q},\frac{1}{2},\frac{qx^{2\beta}}{2\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)(\Gamma(1+\beta))^{2}}\right]),$

Table 6. Invariant solution of subalgebras $\epsilon 1+5$ and $\epsilon 3+5$.

No	*subalgebra*	*invariant*	*solution for Eq.(3.1)*
4	$\epsilon 1+5$	$Z=\frac{X}{\sqrt{T^2+\epsilon}}$ $W=\frac{F(Z)-\frac{qTX}{\sqrt{T^2+\epsilon}}}{\sqrt{T^2+\epsilon}}$	$u(x,t)=((\frac{-q^2px^\beta t^\alpha}{\Gamma(1+\alpha)\Gamma(1+\beta)\sqrt{\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)^{-\frac{1}{2}}+\epsilon}})+(2p^2(C_2\times$ $(\frac{(-q\sqrt{\epsilon}-iC_1)}{(-1)^{-\frac{1}{4}}\sqrt{2q}p\epsilon^{\frac{1}{4}}}\times HermiteH\left[\frac{-3q\sqrt{\epsilon}-iC_1}{2q\sqrt{\epsilon}},\frac{(-e)^{\frac{1}{4}}\sqrt{q}x^\beta}{\sqrt{2\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)}\Gamma(1+\beta)}\right]$ $+\frac{iq\sqrt{\epsilon}x^\beta}{2p\Gamma(1+\beta)}\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)^{-\frac{1}{2}}\times$ $HermiteH\left[\frac{-q\sqrt{\epsilon}-iC_1}{2q\sqrt{\epsilon}},\frac{(-e)^{\frac{1}{4}}\sqrt{q}x^\beta}{\sqrt{2\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)}\Gamma(1+\beta)}\right])$ $-\frac{i(-q\sqrt{\epsilon}-iC_1)x^\beta}{2p\Gamma(1+\beta)}\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)^{-\frac{1}{2}}\times$ $Hypergeometric1F_1\left[\frac{5q\sqrt{\epsilon}+iC_1}{4q\sqrt{\epsilon}},\frac{3}{2},\frac{iq\sqrt{\epsilon}x^{2\beta}(\Gamma(1+\beta))^{-2}}{2\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)}\right]$ $-\frac{iq\sqrt{\epsilon}x^\beta}{2p\Gamma(1+\beta)}\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)^{-\frac{1}{2}}\times$ $Hypergeometric1F_1\left[\frac{q\sqrt{\epsilon}+iC_1}{4q\sqrt{\epsilon}},\frac{1}{2},\frac{iq\sqrt{\epsilon}x^{2\beta}(\Gamma(1+\beta))^{-2}}{2\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)}\right]))$ $/(C_2HermiteH\left[\frac{-q\sqrt{\epsilon}-iC_1}{2q\sqrt{\epsilon}},\frac{(-1)^{\frac{1}{4}}\sqrt{q}x^\beta}{\sqrt{\left(\frac{4qt^\alpha}{\Gamma(1+\alpha)}\right)^2+2\epsilon\Gamma(1+\beta)}}\right]+$ $Hypergeometric1F_1\left[\frac{q\sqrt{\epsilon}+iC_1}{4q\sqrt{\epsilon}},\frac{1}{2},\frac{iq\sqrt{\epsilon}x^{2\beta}(\Gamma(1+\beta))^{-2}}{2\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)}\right]))$ $/\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)^{\frac{1}{2}},$
5	$\epsilon 3+5$	$Z=\frac{X}{\sqrt{T^2+2T\epsilon}}$ $W=\frac{F(Z)-\frac{qTX}{\sqrt{T^2+2T\epsilon}}}{\sqrt{T^2+2T\epsilon}}$	$u(x,t)=\frac{1}{\sqrt{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)}}(\left(\frac{-q^2px^\beta t^\alpha}{\Gamma(1+\alpha)\Gamma(1+\beta)\sqrt{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)}}\right)$ $+((2p^2(C_2(\frac{-q\epsilon x^\beta}{p\Gamma(1+\beta)\sqrt{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)}}\times$ $HermiteH\left[\frac{-2q\epsilon+C_1}{2q\epsilon},\frac{\sqrt{q\epsilon}x^\beta}{\sqrt{\left(\frac{4qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\left(\frac{4q\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)}\Gamma(1+\beta)}\right]+\frac{(-2q\epsilon+C_1)}{p\sqrt{2q\epsilon}}\times$ $HermiteH\left[\frac{-4q\epsilon+C_1}{2q\epsilon},\frac{\sqrt{q\epsilon}x^\beta}{\sqrt{\left(\frac{4qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\left(\frac{4q\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)}\Gamma(1+\beta)}\right])$ $-\frac{q\epsilon x^\beta}{p\Gamma(1+\beta)\sqrt{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)}}\times$ $Hypergeometric1F_1\left[-\frac{-2q\epsilon+C_1}{4q\epsilon},\frac{1}{2},\frac{q\epsilon x^{2\beta}(\Gamma(1+\beta))^{-2}}{2\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)\right)}\right]$ $-\frac{(-2q\epsilon+C_1)x^\beta}{2p\Gamma(1+\beta)\sqrt{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)}}\times$ $Hypergeometric1F_1\left[\frac{6q\epsilon-C_1}{4q\epsilon},\frac{3}{2},\frac{q\epsilon x^{2\beta}(\Gamma(1+\beta))^{-2}}{2\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)\right)}\right]))$

Table 7. Invariant solution of subalgebras $1, 3, 5,\ 1+2$ and $1+4$.

No	*subalgebra*	*invariant*	*solution for Eq.(3.1)*
			$/(C_2 HermiteH\left[\frac{-2q\epsilon+C_1}{2q\epsilon}, \frac{\sqrt{\frac{q\epsilon}{2}}(\Gamma(1+\beta))^{-1}x^{\beta}}{\sqrt{\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)\right)}}\right]+$ $Hypergeometric1F_1\left[\frac{2q\epsilon-C_1}{4q\epsilon}, \frac{1}{2}, \frac{q\epsilon x^{2\beta}(\Gamma(1+\beta))^{-2}}{2\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)\right)}\right]))),$
6	$1, 3, 5$	$Z = XT^{-\frac{3}{2}}$ $W = \frac{\frac{F(Z)}{T}-qZ}{X}$	$u(x,t) = ((\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-1} (\frac{-4p^2px^{\beta}}{\Gamma(1+\beta)}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{3}{2}} (-\frac{\sqrt{q}}{4p}\left(\frac{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{\frac{3}{2}}}{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)}\right)^{\frac{3}{2}}$ $\times Hypergeometric1F_1\left[\frac{q-2C_1}{2q}, \frac{3}{2}, \frac{-q}{\left(\frac{4p^3x^{\beta}}{\Gamma(1+\beta)}\right)\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{3}{2}}}\right]$ $-\frac{C_1C_2}{2p^2\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{3}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)\right)^2}\times$ $Hypergeometric1F_1\left[\frac{q-C_1}{q}, \frac{3}{2}, \frac{-q}{\left(\frac{4p^3x^{\beta}}{\Gamma(1+\beta)}\right)\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{3}{2}}}\right]$ $+\frac{q^{\frac{3}{2}}(q-2C_1)}{12p^3 2q}\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{\frac{3}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)^{-1}\right)^{\frac{5}{2}}\times$ $Hypergeometric1F_1\left[\frac{3q-2C_1}{2q}, \frac{5}{2}, \frac{-q}{\left(\frac{4p^3x^{\beta}}{\Gamma(1+\beta)}\right)\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{3}{2}}}\right]))/$ $(C_2 Hypergeometric1F_1\left[\frac{-C_1}{q}, \frac{1}{2}, \frac{-q}{\left(\frac{4p^3x^{\beta}}{\Gamma(1+\beta)}\right)\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{3}{2}}}\right]$ $+\frac{\sqrt{\frac{q}{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{3}{2}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)}}}{2p}\times$ $Hypergeometric1F_1\left[\frac{q-2C_1}{2q}, \frac{3}{2}, \frac{-q}{\left(\frac{4p^3px^{\beta}}{\Gamma(1+\beta)}\right)\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{3}{2}}}\right])$ $-\frac{q\Gamma(1+\beta)}{px^{\beta}}\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{\frac{3}{2}}\left(\frac{\Gamma(1+\beta)}{px^{\beta}}\right)),$
7	$1+2$	$Z = X - T$ $W = F(Z)$	$u(x,t) = -q + \sqrt{-q^2-2p^2C_1}\times$ $Tan\left[\frac{-\sqrt{-q^2-2p^2C_1}\left(\left(\frac{px^{\beta}}{\Gamma(1+\beta)}\right)-\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)-\sqrt{-q^2-2p^2C_1}C_2}{2p^2}\right],$
8	$1+4$	$Z = -X + \frac{T^2}{2}$ $W = -qT + F(Z)$	$u(x,t) = -\frac{2q^2}{2^{\frac{1}{3}}p^2\left(\frac{q^2}{p^4}\right)^{\frac{2}{3}}}\times$ $(AiryBi\Pr ime\left[\frac{2^{\frac{2}{3}}p^{\frac{8}{3}}}{q^{\frac{4}{3}}}\left(\frac{q^4p^{-4}t^{2\alpha}}{4(\Gamma(1+\alpha))^2}-\frac{q^2p^{-3}x^{\beta}}{2\Gamma(1+\beta)}+\frac{C_1}{2p^2}\right)\right]$ $+C_2\times$ $AiryAi\Pr ime\left[\frac{2^{\frac{2}{3}}p^{\frac{8}{3}}}{q^{\frac{4}{3}}}\left(\frac{q^4p^{-4}t^{2\alpha}}{4(\Gamma(1+\alpha))^2}-\frac{q^2p^{-3}x^{\beta}}{2\Gamma(1+\beta)}+\frac{C_1}{2p^2}\right)\right])$ $/(AiryBi\left[\frac{2^{\frac{2}{3}}p^{\frac{8}{3}}}{q^{\frac{4}{3}}}\left(\frac{q^4p^{-4}t^{2\alpha}}{4(\Gamma(1+\alpha))^2}-\frac{q^2p^{-3}x^{\beta}}{2\Gamma(1+\beta)}+\frac{C_1}{2p^2}\right)\right]$ $+C_2 AiryAi\left[\frac{2^{\frac{2}{3}}p^{\frac{8}{3}}}{q^{\frac{4}{3}}}\left(\frac{q^4p^{-4}t^{2\alpha}}{4(\Gamma(1+\alpha))^2}-\frac{q^2p^{-3}x^{\beta}}{2\Gamma(1+\beta)}+\frac{C_1}{2p^2}\right)\right])-\frac{q^2t^{\alpha}}{\Gamma(1+\alpha)},$

Table 8. Invariant solution of subalgebras $1+2+4$, $2+5$, $4+5$ and $2+4+5$.

No	*subalgebra*	*invariant*	*solution for Eq.(3.1)*
9	$1+2+4$	$Z=-X+\frac{T^2}{2}+T$ $W=-qT+F(Z)$	$u(x,t)=(-2p^2(\frac{q}{2p^2}\times$ $AiryBi\left[\frac{q^{\frac{5}{3}}p^{-\frac{4}{3}}t^{\alpha}}{2^{\frac{1}{3}}\Gamma(1+\alpha)}+\frac{q^{\frac{8}{3}}p^{-\frac{4}{3}}t^{2\alpha}}{2^{\frac{4}{3}}(\Gamma(1+\alpha))^2}-\frac{q^{\frac{2}{3}}p^{-\frac{1}{3}}x^{\beta}}{2^{\frac{1}{3}}\Gamma(1+\beta)}+\frac{q^2+2p^2C_1}{2^{\frac{4}{3}}p^{\frac{4}{3}}q^{\frac{4}{3}}}\right]+\frac{q^{\frac{2}{3}}}{2^{\frac{1}{3}}p^{\frac{4}{3}}}\times$ $AiryBiPrime\left[\frac{q^{\frac{5}{3}}p^{-\frac{4}{3}}t^{\alpha}}{2^{\frac{1}{3}}\Gamma(1+\alpha)}+\frac{q^{\frac{8}{3}}p^{-\frac{4}{3}}t^{2\alpha}}{2^{\frac{4}{3}}(\Gamma(1+\alpha))^2}-\frac{q^{\frac{2}{3}}p^{-\frac{1}{3}}x^{\beta}}{2^{\frac{1}{3}}\Gamma(1+\beta)}+\frac{q^2+2p^2C_1}{2^{\frac{4}{3}}p^{\frac{4}{3}}q^{\frac{4}{3}}}\right]$ $+C_2(\frac{q}{2p^2}\times AiryAi\left[\frac{q^{\frac{5}{3}}p^{-\frac{4}{3}}t^{\alpha}}{2^{\frac{1}{3}}\Gamma(1+\alpha)}+\frac{q^{\frac{8}{3}}p^{-\frac{4}{3}}t^{2\alpha}}{2^{\frac{4}{3}}(\Gamma(1+\alpha))^2}-\frac{q^{\frac{2}{3}}p^{-\frac{1}{3}}x^{\beta}}{2^{\frac{1}{3}}\Gamma(1+\beta)}+\frac{q^2+2p^2C_1}{2^{\frac{4}{3}}p^{\frac{4}{3}}q^{\frac{4}{3}}}\right]$ $+\frac{q^{\frac{2}{3}}}{2^{\frac{1}{3}}p^{\frac{4}{3}}}\times$ $AiryAiPrime\left[\frac{q^{\frac{5}{3}}p^{-\frac{4}{3}}t^{\alpha}}{2^{\frac{1}{3}}\Gamma(1+\alpha)}+\frac{q^{\frac{8}{3}}p^{-\frac{4}{3}}t^{2\alpha}}{2^{\frac{4}{3}}(\Gamma(1+\alpha))^2}-\frac{q^{\frac{2}{3}}p^{-\frac{1}{3}}x^{\beta}}{2^{\frac{1}{3}}\Gamma(1+\beta)}+\frac{q^2+2p^2C_1}{2^{\frac{4}{3}}p^{\frac{4}{3}}q^{\frac{4}{3}}}\right]))$ $/(AiryBi\left[\frac{q^{\frac{5}{3}}p^{-\frac{4}{3}}t^{\alpha}}{2^{\frac{1}{3}}\Gamma(1+\alpha)}+\frac{q^{\frac{8}{3}}p^{-\frac{4}{3}}t^{2\alpha}}{2^{\frac{4}{3}}(\Gamma(1+\alpha))^2}-\frac{q^{\frac{2}{3}}p^{-\frac{1}{3}}x^{\beta}}{2^{\frac{1}{3}}\Gamma(1+\beta)}+\frac{q^2+2p^2C_1}{2^{\frac{4}{3}}p^{\frac{4}{3}}q^{\frac{4}{3}}}\right]$ $+C_2AiryAi\left[\frac{q^{\frac{5}{3}}p^{-\frac{4}{3}}t^{\alpha}}{2^{\frac{1}{3}}\Gamma(1+\alpha)}+\frac{q^{\frac{8}{3}}p^{-\frac{4}{3}}t^{2\alpha}}{2^{\frac{4}{3}}(\Gamma(1+\alpha))^2}-\frac{q^{\frac{2}{3}}p^{-\frac{1}{3}}x^{\beta}}{2^{\frac{1}{3}}\Gamma(1+\beta)}+\frac{q^2+2p^2C_1}{2^{\frac{4}{3}}p^{\frac{4}{3}}q^{\frac{4}{3}}}\right])$ $-\frac{q^2t^{\alpha}}{\Gamma(1+\alpha)},$
10	$2+5$	$Z=\frac{2TX+1}{2T^2}$ $W=\frac{2TF(Z)-q}{2T^2}-qZ$	$u(x,t)=\frac{2p^{\frac{2}{3}}\Gamma(1+\alpha)}{2^{\frac{1}{3}}q^{\frac{1}{3}}t^{\alpha}}\times$ $(AiryBiPrime\left[\frac{2^{\frac{2}{3}}p^{\frac{8}{3}}}{q^{\frac{4}{3}}}\left(\frac{qx^{\beta}\Gamma(1+\alpha)}{2p^3t^{\alpha}\Gamma(1+\beta)}+\frac{(\Gamma(1+\alpha))^2}{4p^4t^{2\alpha}}-\frac{C_1}{2p^2}\right)\right]$ $+C_2AiryAiPrime\left[\frac{2^{\frac{2}{3}}p^{\frac{8}{3}}}{q^{\frac{4}{3}}}\left(\frac{qx^{\beta}\Gamma(1+\alpha)}{2p^3t^{\alpha}\Gamma(1+\beta)}+\frac{(\Gamma(1+\alpha))^2}{4p^4t^{2\alpha}}-\frac{C_1}{2p^2}\right)\right])$ $/(AiryBi\left[\frac{2^{\frac{2}{3}}p^{\frac{8}{3}}}{q^{\frac{4}{3}}}\left(\frac{qx^{\beta}\Gamma(1+\alpha)}{2p^3t^{\alpha}\Gamma(1+\beta)}+\frac{(\Gamma(1+\alpha))^2}{4p^4t^{2\alpha}}-\frac{C_1}{2p^2}\right)\right]$ $+C_2AiryAi\left[\frac{2^{\frac{2}{3}}p^{\frac{8}{3}}}{q^{\frac{4}{3}}}\left(\frac{qx^{\beta}\Gamma(1+\alpha)}{2p^3t^{\alpha}\Gamma(1+\beta)}+\frac{(\Gamma(1+\alpha))^2}{4p^4t^{2\alpha}}-\frac{C_1}{2p^2}\right)\right])$ $-\frac{px^{\beta}\Gamma(1+\alpha)}{t^{\alpha}\Gamma(1+\beta)}-\frac{(\Gamma(1+\alpha))^2}{qt^{2\alpha}},$
11	$4+5$	$Z=\frac{X+1}{T}$ $W=-qZ+\frac{F(Z)}{T}$	$u(x,t)=-\frac{\Gamma(1+\alpha)}{t^{\alpha}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)+\frac{\sqrt{2C_1}p\Gamma(1+\alpha)}{qt^{\alpha}}\times$ $Tanh\left[\frac{\sqrt{2C_1}\left(\frac{\Gamma(1+\alpha)}{qt^{\alpha}}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)\right)+\sqrt{2C_1}C_2}{2p}\right],$
12	$2+4+5$	$Z=\frac{X+1}{T}+\frac{1}{2T^2}$ $W=\frac{2TF(Z)-q}{2T^2}-qZ$	$u(x,t)=\frac{2^{\frac{2}{3}}p^{\frac{2}{3}}\Gamma(1+\alpha)}{q^{\frac{1}{3}}t^{\alpha}}\times($ $AiryBiPrime\left[\frac{q^{-\frac{1}{3}}x^{\beta}\Gamma(1+\alpha)}{2^{\frac{1}{3}}p^{\frac{1}{3}}t^{\alpha}\Gamma(1+\beta)}+\frac{q^{-\frac{1}{3}}\Gamma(1+\alpha)}{2^{\frac{1}{3}}p^{\frac{4}{3}}t^{\alpha}}+\frac{(\Gamma(1+\alpha))^2}{2^{\frac{4}{3}}(pq)^{\frac{4}{3}}t^{2\alpha}}-\frac{p^{\frac{2}{3}}C_1}{2^{\frac{1}{3}}q^{\frac{4}{3}}}\right]$ $+C_2\times$ $AiryAiPrime\left[\frac{q^{-\frac{1}{3}}x^{\beta}\Gamma(1+\alpha)}{2^{\frac{1}{3}}p^{\frac{1}{3}}t^{\alpha}\Gamma(1+\beta)}+\frac{q^{-\frac{1}{3}}\Gamma(1+\alpha)}{2^{\frac{1}{3}}p^{\frac{4}{3}}t^{\alpha}}+\frac{(\Gamma(1+\alpha))^2}{2^{\frac{4}{3}}(pq)^{\frac{4}{3}}t^{2\alpha}}-\frac{p^{\frac{2}{3}}C_1}{2^{\frac{1}{3}}q^{\frac{4}{3}}}\right])$ $/(AiryBi\left[\frac{q^{-\frac{1}{3}}x^{\beta}\Gamma(1+\alpha)}{2^{\frac{1}{3}}p^{\frac{1}{3}}t^{\alpha}\Gamma(1+\beta)}+\frac{q^{-\frac{1}{3}}\Gamma(1+\alpha)}{2^{\frac{1}{3}}p^{\frac{4}{3}}t^{\alpha}}+\frac{(\Gamma(1+\alpha))^2}{2^{\frac{4}{3}}(pq)^{\frac{4}{3}}t^{2\alpha}}-\frac{p^{\frac{2}{3}}C_1}{2^{\frac{1}{3}}q^{\frac{4}{3}}}\right]$ $+C_2AiryAi\left[\frac{q^{-\frac{1}{3}}x^{\beta}\Gamma(1+\alpha)}{2^{\frac{1}{3}}p^{\frac{1}{3}}t^{\alpha}\Gamma(1+\beta)}+\frac{q^{-\frac{1}{3}}\Gamma(1+\alpha)}{2^{\frac{1}{3}}p^{\frac{4}{3}}t^{\alpha}}+\frac{(\Gamma(1+\alpha))^2}{2^{\frac{4}{3}}(pq)^{\frac{4}{3}}t^{2\alpha}}-\frac{p^{\frac{2}{3}}C_1}{2^{\frac{1}{3}}q^{\frac{4}{3}}}\right])$ $-\frac{(\Gamma(1+\alpha))^2}{qt^{2\alpha}}-\frac{px^{\beta}\Gamma(1+\alpha)}{t^{\alpha}\Gamma(1+\beta)}-\frac{\Gamma(1+\alpha)}{t^{\alpha}},$

Table 9. Invariant solution of subalgebras $2+3$, $3+4$ and $2+3+4$.

No	subalgebra	invariant	solution for Eq.(3.1)
13	$2+3$	$Z=\frac{X}{\sqrt{2T+1}}$ $W=\frac{F(Z)}{\sqrt{2T+1}}$	$u(x,t)=\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)^{-\frac{1}{2}}(2p^{2}(C_{2}(-\frac{qx^{\beta}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)^{-\frac{1}{2}}}{p\Gamma(1+\beta)}$ $HermiteH\left[\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}x^{\beta}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)^{-\frac{1}{2}}}{\sqrt{2}\Gamma(1+\beta)}\right]$ $+\frac{(-2q+C_{1})}{p\sqrt{2q}}HermiteH\left[-\frac{4q+C_{1}}{2q},\frac{\sqrt{q}x^{\beta}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)^{-\frac{1}{2}}}{\sqrt{2}\Gamma(1+\beta)}\right])$ $-\frac{qx^{\beta}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)^{-\frac{1}{2}}}{p\Gamma(1+\beta)}Hypergeometric1F_{1}\left[\frac{2q-C_{1}}{4q},\frac{1}{2},\frac{q\left(\frac{x^{\beta}}{\Gamma(1+\beta)}\right)^{2}}{2\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)}\right]$ $+\frac{(2q-C_{1})x^{\beta}}{2p\Gamma(1+\beta)\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1}}Hypergeometric1F_{1}\left[-\frac{6q+C_{1}}{4q},\frac{3}{2},\frac{q\left(\frac{x^{\beta}}{\Gamma(1+\beta)}\right)^{2}}{2\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)}\right]))$ $/(C_{2}HermiteH\left[\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}x^{\beta}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)^{-\frac{1}{2}}}{\sqrt{2}\Gamma(1+\beta)}\right]+$ $Hypergeometric1F_{1}\left[\frac{2q-C_{1}}{4q},\frac{1}{2},\frac{q\left(\frac{x^{\beta}}{\Gamma(1+\beta)}\right)^{2}}{2\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)}\right]),$
14	$3+4$	$Z=\frac{(X-T)}{\sqrt{T}}$ $W=\frac{F(Z)}{\sqrt{T}}-q$	$u(x,t)=-q+\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}(2p^{2}(C_{2}(-\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)}{2p^{2}\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\times$ $HermiteH\left[\frac{-q+C_{1}}{q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)}{2p\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\right]+\frac{(-q+C_{1})}{p\sqrt{q}}$ $HermiteH\left[-1+\frac{-q+C_{1}}{q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)}{2p\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\right])-\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)}{2p^{2}\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}$ $Hypergeometric1F_{1}\left[\frac{q-C_{1}}{2q},\frac{1}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}}{4p^{2}\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]-\frac{(-q+C_{1})}{2p^{2}\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\times$ $\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)Hypergeometric1F_{1}\left[1+\frac{q-C_{1}}{2q},\frac{3}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}}{4p^{2}\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]))$ $/(C_{2}HermiteH\left[\frac{-q+C_{1}}{q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)}{2p\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\right]+$ $Hypergeometric1F_{1}\left[\frac{q-C_{1}}{2q},\frac{1}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}}{4p^{2}\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]),$
15	$2+3+4$	$Z=\frac{X+1-T}{\sqrt{T}}$ $W=\frac{F(Z)}{\sqrt{T}}-q$	$u(x,t)=-q+\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}(2p^{2}(C_{2}(-\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)}{2p^{2}\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\times$ $HermiteH\left[\frac{-q+C_{1}}{q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)}{2p\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\right]$ $+\frac{(-q+C_{1})}{p\sqrt{q}}HermiteH\left[\frac{-2q+C_{1}}{q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)}{2p\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\right])$

Table 10. Invariant solution of subalgebras $\epsilon 1+2+3$ and $\epsilon 1+3+4$

No	$subalgebra$	$invariant$	$solution\ for\ Eq.(3.1)$
			$-\frac{\frac{pqx^{\beta}}{\Gamma(1+\beta)}-\frac{q^{2}t^{\alpha}}{\Gamma(1+\alpha)}+q}{2p^{2}\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}Hypergeometric1F_{1}\left[\frac{q-C_{1}}{2q},\frac{1}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)^{2}}{\left(\frac{4p^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]$ $+\frac{(q-C_{1})\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)}{2p^{2}\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\times$ $Hypergeometric1F_{1}\left[\frac{-3q+C_{1}}{2q},\frac{3}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)^{2}}{\left(\frac{4p^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]))$ $/(C_{2}HermiteH\left[\frac{-q+C_{1}}{q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)}{2p\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}}\right]+$ $Hypergeometric1F_{1}\left[\frac{q-C_{1}}{2q},\frac{1}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+1\right)^{2}}{\left(\frac{4p^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]),$
16	$\epsilon 1+2+3$	$Z=\frac{X+1}{\sqrt{2T+\epsilon}}$ $W=\frac{F(Z)}{\sqrt{2T+\epsilon}}$	$u\,(x,t)=\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)^{-\frac{1}{2}}\ (2p^{2}(C_{2}(-\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{p^{2}\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}$ $\times HermiteH\left[\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{\sqrt{2}p\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}\right]+\frac{(-2q+C_{1})}{p\sqrt{2q}}$ $HermiteH\left[-1+\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{\sqrt{2}p\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}\right])-\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{p^{2}\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}$ $Hypergeometric1F_{1}\left[\frac{2q-C_{1}}{4q},\frac{1}{2},\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)^{2}}{2p^{2}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)}\right]-\frac{(-2q+C_{1})}{2p^{2}\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}$ $\times\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)Hypergeometric1F_{1}\left[\frac{6q-C_{1}}{4q},\frac{3}{2},\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)^{2}}{\left(\frac{4qp^{2}t^{\alpha}}{\Gamma(1+\alpha)}+2p^{2}\epsilon\right)}\right]))$ $/(C_{2}HermiteH\left[\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{\sqrt{2}p\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}\right]$ $+Hypergeometric1F_{1}\left[\frac{2q-C_{1}}{4q},\frac{1}{2},\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)^{2}}{2p^{2}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)}\right]),$
17	$\epsilon 1+3+4$	$Z=\frac{X-T-\epsilon}{\sqrt{2T+\epsilon}}$ $W=\frac{F(Z)}{\sqrt{2T+\epsilon}}-q$	$u\,(x,t)=-q+\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)^{-\frac{1}{2}}\ (2p^{2}(C_{2}(-\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}-\epsilon\right)}{p^{2}\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}$ $\times HermiteH\left[\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}-\epsilon\right)}{\sqrt{2}p\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}\right]+\frac{(-2q+C_{1})}{p\sqrt{2q}}$ $HermiteH\left[-1+\frac{-2q+C_{1}}{2q},\frac{\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}-\epsilon\right)}{\sqrt{2}p\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}\right])-\frac{\frac{pqx^{\beta}}{\Gamma(1+\beta)}-\frac{q^{2}t^{\alpha}}{\Gamma(1+\alpha)}-q\epsilon}{p^{2}\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}$ $Hypergeometric1F_{1}\left[\frac{2q-C_{1}}{4q},\frac{1}{2},\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}-\epsilon\right)^{2}}{2p^{2}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)}\right]-\frac{(-2q+C_{1})}{2p^{2}\sqrt{\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon}}\times$ $\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}-\epsilon\right)\times$ $Hypergeometric1F_{1}\left[1-\frac{-2q+C_{1}}{4q},\frac{3}{2},\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}-\epsilon\right)^{2}}{2p^{2}\left(\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}+\epsilon\right)}\right]))$

Table 11. Invariant solution of subalgebras $\epsilon 1+2+3+4$ and $\epsilon 1+2+5$

No	*subalgebra*	*invariant*	*solution for Eq.(3.1)*
			$/(C_2 HermiteH\left[\frac{-2q+C_1}{2q}, \frac{\sqrt{q}\left(\frac{px^\beta}{\Gamma(1+\beta)}-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)}{\sqrt{2}p\sqrt{\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon}}\right]$ $+Hypergeometric1F_1\left[\frac{2q-C_1}{4q}, \frac{1}{2}, \frac{q\left(\frac{px^\beta}{\Gamma(1+\beta)}-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)^2}{2p^2\left(\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon\right)}\right]),$
18	$\epsilon 1+2+3+4$	$Z=\frac{X+1-T-\epsilon}{\sqrt{2T+\epsilon}}$ $W=\frac{F(Z)}{\sqrt{2T+\epsilon}}-q$	$u(x,t)=-q+\left(\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon\right)^{-\frac{1}{2}}(2p^2(C_2(-\frac{q\left(\frac{px^\beta}{\Gamma(1+\beta)}+1-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)}{p^2\sqrt{\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon}}$ $\times HermiteH\left[\frac{-2q+C_1}{2q}, \frac{\sqrt{q}\left(\frac{px^\beta}{\Gamma(1+\beta)}+1-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)}{\sqrt{2}p\sqrt{\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon}}\right]+\frac{(-2q+C_1)}{p\sqrt{2q}}$ $HermiteH\left[\frac{-4q+C_1}{2q}, \frac{\left(\frac{px^\beta}{\Gamma(1+\beta)}+1-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)}{\sqrt{2}pq^{-\frac{1}{2}}\sqrt{\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon}}\right])-\frac{\left(\frac{px^\beta}{\Gamma(1+\beta)}+1-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)}{p^2\sqrt{\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon}}$ $\times qHypergeometric1F_1\left[\frac{2q-C_1}{4q}, \frac{1}{2}, \frac{q\left(\frac{px^\beta}{\Gamma(1+\beta)}+1-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)^2}{2p^2\left(\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon\right)}\right]$ $-\frac{(-2q+C_1)\left(\frac{px^\beta}{\Gamma(1+\beta)}+1-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)}{2p^2\sqrt{\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon}})\times$ $Hypergeometric1F_1\left[\frac{6q-C_1}{4q}, \frac{3}{2}, \frac{q\left(\frac{px^\beta}{\Gamma(1+\beta)}+1-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)^2}{2p^2\left(\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon\right)}\right])$ $/(C_2 HermiteH\left[\frac{-2q+C_1}{2q}, \frac{\sqrt{q}\left(\frac{px^\beta}{\Gamma(1+\beta)}+1-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)}{\sqrt{2}p\sqrt{\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon}}\right]+$ $Hypergeometric1F_1\left[\frac{2q-C_1}{4q}, \frac{1}{2}, \frac{q\left(\frac{px^\beta}{\Gamma(1+\beta)}+1-\frac{qt^\alpha}{\Gamma(1+\alpha)}-\epsilon\right)^2}{2p^2\left(\frac{2qt^\alpha}{\Gamma(1+\alpha)}+\epsilon\right)}\right]),$
19	$\epsilon 1+2+5$	$Z=\frac{X-\frac{T}{\epsilon}}{\sqrt{T^2+\epsilon}}$ $W=\frac{F(Z)-qTZ}{\sqrt{2T+\epsilon}}-\frac{q}{\epsilon}$	$u(x,t)=\epsilon^{-1}\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)^{-\frac{1}{2}}(-q(\left(\frac{\epsilon qt^\alpha}{\Gamma(1+\alpha)}\right)\times$ $\left(\frac{px^\beta}{\Gamma(1+\beta)}-\frac{qt^\alpha}{\epsilon\Gamma(1+\alpha)}\right)\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)^{-\frac{1}{2}}$ $+\left(\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)^{-\frac{1}{2}}+\epsilon\right)^{-\frac{1}{2}})+\epsilon(2p^2(C_2(\frac{(-1)^{\frac{1}{4}}(-q\sqrt{\epsilon}-iC_1)}{p\sqrt{2q}\epsilon^{\frac{1}{4}}}$ $\times HermiteH\left[\frac{-3q\sqrt{\epsilon}-iC_1}{2q\sqrt{\epsilon}}, \frac{(-e)^{\frac{1}{4}}\sqrt{q}\left(\frac{px^\beta}{\Gamma(1+\beta)}-\frac{qt^\alpha}{\epsilon\Gamma(1+\alpha)}\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon}}\right]$ $+\frac{iq\left(\frac{px^\beta}{\Gamma(1+\beta)}-\frac{qt^\alpha}{\epsilon\Gamma(1+\alpha)}\right)\sqrt{\epsilon}}{2p^2\sqrt{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon}}\times$ $HermiteH\left[\frac{-q\sqrt{\epsilon}-iC_1}{2q\sqrt{\epsilon}}, \frac{(-e)^{\frac{1}{4}}\sqrt{q}\left(\frac{px^\beta}{\Gamma(1+\beta)}-\frac{qt^\alpha}{\epsilon\Gamma(1+\alpha)}\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon}}\right])$ $-\frac{i\left(\frac{px^\beta}{\Gamma(1+\beta)}-\frac{qt^\alpha}{\epsilon\Gamma(1+\alpha)}\right)(-q\sqrt{\epsilon}-iC_1)}{2p^2\sqrt{\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon}}\times$ $Hypergeometric1F_1\left[\frac{-5q\sqrt{\epsilon}-iC_1}{4q\sqrt{\epsilon}}, \frac{3}{2}, \frac{iq\left(\frac{px^\beta}{\Gamma(1+\beta)}-\frac{qt^\alpha}{\epsilon\Gamma(1+\alpha)}\right)^2\sqrt{\epsilon}}{2p^2\left(\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)}\right]$

Table 12. Invariant solution of subalgebras $\epsilon 1+4+5$ and $\epsilon 1+2+4+5$

No	$subalgebra$	$invariant$	$solution\ for\ Eq.(3.1)$
			$-\frac{iq\sqrt{\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)}{2p^{2}\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}\times$ $Hypergeometric1F_{1}\left[\frac{q\sqrt{\epsilon}+iC_{1}}{4q\sqrt{\epsilon}},\frac{1}{2},\frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)^{2}\sqrt{\epsilon}}{2p^{2}\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon\right)}\right]))/$ $(C_{2}HermiteH\left[\frac{-q\sqrt{\epsilon}-iC_{1}}{2q\sqrt{\epsilon}},\frac{(-1)^{\frac{1}{4}}\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}\right]$ $+Hypergeometric1F_{1}\left[\frac{q\sqrt{\epsilon}+iC_{1}}{4q\sqrt{\epsilon}},\frac{1}{2},\frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)^{2}\sqrt{\epsilon}}{2p^{2}\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon\right)}\right])),$
20	$\epsilon 1+4+5$	$Z=\frac{X+1}{\sqrt{T^{2}+\epsilon}}$ $W=\frac{F(Z)-qTZ}{\sqrt{T^{2}+\epsilon}}$	$u(x,t)=\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon\right)^{-\frac{1}{2}}(-q(\frac{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}})$ $+(2p^{2}(C_{2}(\frac{(-1)^{\frac{1}{4}}(-q\sqrt{\epsilon}-iC_{1})}{p\sqrt{2q}\epsilon^{\frac{1}{4}}}\times$ $HermiteH\left[\frac{-3q\sqrt{\epsilon}-iC_{1}}{2q\sqrt{\epsilon}},\frac{(-e)^{\frac{1}{4}}\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}\right]$ $+\frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)\sqrt{\epsilon}}{2p^{2}\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}HermiteH\left[\frac{-q\sqrt{\epsilon}-iC_{1}}{2q\sqrt{\epsilon}},\frac{(-e)^{\frac{1}{4}}\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}\right])$ $-\frac{i\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)(-q\sqrt{\epsilon}-iC_{1})}{2p^{2}\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}\times$ $Hypergeometric1F_{1}\left[\frac{5q\sqrt{\epsilon}+iC_{1}}{4q\sqrt{\epsilon}},\frac{3}{2},\frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)^{2}\sqrt{\epsilon}}{2p^{2}\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon\right)}\right]$ $-\frac{iq\sqrt{\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{2p^{2}\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}\times$ $Hypergeometric1F_{1}\left[\frac{q\sqrt{\epsilon}+iC_{1}}{4q\sqrt{\epsilon}},\frac{1}{2},\frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)^{2}\sqrt{\epsilon}}{2p^{2}\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon\right)}\right]))$ $/(C_{2}HermiteH\left[\frac{-q\sqrt{\epsilon}-iC_{1}}{2q\sqrt{\epsilon}},\frac{(-1)^{\frac{1}{4}}\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}\right]$ $+Hypergeometric1F_{1}\left[\frac{q\sqrt{\epsilon}+iC_{1}}{4q\sqrt{\epsilon}},\frac{1}{2},\frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+1\right)^{2}\sqrt{\epsilon}}{2p^{2}\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon\right)}\right])),$
21	$\epsilon 1+2+4+5$	$Z=\frac{X-\frac{T-\epsilon}{\epsilon}}{\sqrt{T^{2}+\epsilon}}$ $W=\frac{F(Z)-qTZ}{\sqrt{T^{2}+\epsilon}}-\frac{q}{\epsilon}$	$u(x,t)=\frac{\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}{\epsilon}(-q(\frac{\left(\frac{\epsilon qt^{\alpha}}{\Gamma(1+\alpha)}\right)\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)}{\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}$ $+\left(\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon\right)^{-\frac{1}{2}}+\epsilon\right)^{-\frac{1}{2}})$ $+\epsilon(2p^{2}(C_{2}(\frac{(-1)^{\frac{1}{4}}(-q\sqrt{\epsilon}-iC_{1})}{p\sqrt{2q}\epsilon^{\frac{1}{4}}}\times$ $HermiteH\left[\frac{-3q\sqrt{\epsilon}-iC_{1}}{2q\sqrt{\epsilon}},\frac{(-e)^{\frac{1}{4}}\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\epsilon}}\right]$

Table 13.invariant solution of a subalgebra $\epsilon 3+2+5$

No	subalgebra	invariant	solution for Eq.(3.1)
			$+\frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)}{2p^2\epsilon^{-\frac{1}{2}}\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\epsilon}}\times$ $HermiteH\left[\frac{-q\sqrt{\epsilon}-iC_1}{2q\sqrt{\epsilon}}, \frac{(-e)^{\frac{1}{4}}\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\epsilon}}\right])$ $-\frac{i\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)(-q\sqrt{\epsilon}-iC_1)}{2p^2\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\epsilon}}\times$ $Hypergeometric1F_1\left[\frac{-5q\sqrt{\epsilon}-iC_1}{4q\sqrt{\epsilon}}, \frac{3}{2}, \frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)^2}{2p^2\epsilon^{-\frac{1}{2}}\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)}\right]$ $-\frac{iq\sqrt{\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)}{2p^2\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\epsilon}}\times$ $Hypergeometric1F_1\left[\frac{q\sqrt{\epsilon}+iC_1}{4q\sqrt{\epsilon}}, \frac{1}{2}, \frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)^2}{2p^2\epsilon^{-\frac{1}{2}}\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)}\right]))$ $/(C_2 HermiteH\left[\frac{-q\sqrt{\epsilon}-iC_1}{2q\sqrt{\epsilon}}, \frac{(-1)^{\frac{1}{4}}\sqrt{q}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\epsilon}}\right]$ $+Hypergeometric1F_1\left[\frac{q\sqrt{\epsilon}+iC_1}{4q\sqrt{\epsilon}}, \frac{1}{2}, \frac{iq\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}-1\right)^2}{2p^2\epsilon^{-\frac{1}{2}}\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\epsilon\right)}\right])),$
22	$\epsilon 3+2+5$	$Z=\frac{X+\frac{T+\epsilon}{\epsilon^2}}{\sqrt{T^2+2T\epsilon}}$ $W=\frac{\epsilon^2F(Z)+qT+2q\epsilon}{\epsilon^2\sqrt{T^2+2T\epsilon}}$ $-qZ$	$u(x,t)=\frac{\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}}{\epsilon^2}((\frac{q^2t^{\alpha}}{\Gamma(1+\alpha)}+\epsilon q(2-$ $\frac{\epsilon\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)}{\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}}))+\epsilon^2((2p^2(C_2(-\frac{q\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)\epsilon}{p^2\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}}$ $HermiteH\left[\frac{-2q\epsilon+C_1}{2q\epsilon}, \frac{\sqrt{q\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}}\right]$ $+\frac{(-2q\epsilon+C_1)}{p\sqrt{2q\epsilon}}HermiteH\left[\frac{-4q\epsilon+C_1}{2q\epsilon}, \frac{\sqrt{q\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}}\right])$ $-\frac{q\epsilon\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)}{p^2\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}}\times$ $Hypergeometric1F_1\left[\frac{2q\epsilon-C_1}{4q\epsilon}, \frac{1}{2}, \frac{q\epsilon\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)^2}{2p^2\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)\right)}\right]$ $-\frac{(-2q\epsilon+C_1)\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)}{2p^2\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}}\times$ $Hypergeometric1F_1\left[\frac{6q\epsilon-C_1}{4q\epsilon}, \frac{3}{2}, \frac{q\epsilon\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)^2}{2p^2\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}\right]))$ $/(C_2 HermiteH\left[\frac{-2q\epsilon+C_1}{2q\epsilon}, \frac{\sqrt{q\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}}\right]+$ $Hypergeometric1F_1\left[\frac{2q\epsilon-C_1}{4q\epsilon}, \frac{1}{2}, \frac{q\epsilon\left(\frac{px^{\beta}}{\Gamma(1+\beta)}+\frac{qt^{\alpha}\epsilon}{\epsilon^2\Gamma(1+\alpha)}\right)^2}{2p^2\left(\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^2+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)\right)}\right]))),$

Table 14. Invariant solution of subalgebras $\epsilon 3+4+5$ and $\epsilon 3+2+4+5$

No	*subalgebra*	*invariant*	*solution for Eq.(3.1)*
23	$\epsilon 3+4+5$	$Z=\frac{X-\frac{T}{\epsilon}}{\sqrt{T^{2}+2T\epsilon}}$ $W=\frac{(T\epsilon+2\epsilon^{2})F(Z)}{\sqrt{T^{2}+2T\epsilon}}$ $-\frac{T+2\epsilon+X\epsilon-T\epsilon}{q^{-1}(T+2\epsilon)}$	$u(x,t)=\frac{1}{\frac{q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}+2\epsilon^{2}}((\frac{q^{2}t^{\alpha}}{\Gamma(1+\alpha)}-2q\epsilon-\frac{pq\epsilon x^{\beta}}{\Gamma(1+\beta)}+\frac{q^{2}t^{\alpha}}{\Gamma(1+\alpha)})$ $+\frac{1}{\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}}}((2p^{2}(C_{2}(-\frac{qp^{-2}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)}{\sqrt{\left(\frac{qt^{\alpha}}{\sqrt{\epsilon}\Gamma(1+\alpha)}\right)^{2}+\frac{2qt^{\alpha}}{\Gamma(1+\alpha)}}}\times$ $HermiteH\left[\frac{-2q\epsilon+C_{1}}{2q\epsilon},\frac{\sqrt{q\epsilon}\left(\frac{x^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon p\Gamma(1+\alpha)}\right)}{\sqrt{\left(\frac{\sqrt{2}qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{4q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}}}\right]$ $+\frac{(-2q\epsilon+C_{1})}{p\sqrt{2q\epsilon}}HermiteH\left[\frac{-4q\epsilon+C_{1}}{2q\epsilon},\frac{\sqrt{q\epsilon}\left(\frac{x^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon p\Gamma(1+\alpha)}\right)}{\sqrt{\left(\frac{\sqrt{2}qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{4q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}}}\right])$ $-\frac{q\left(\frac{p\epsilon x^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)}{p^{2}\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}}}\times$ $Hypergeometric1F_{1}\left[\frac{2q\epsilon-C_{1}}{4q\epsilon},\frac{1}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)^{2}}{\left(\frac{\sqrt{2}qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{4p^{2}t^{\alpha}}{\Gamma(1+\alpha)}}\right]$ $+\frac{(2q\epsilon-C_{1})\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)}{2p^{2}\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}}}\times$ $Hypergeometric1F_{1}\left[\frac{6q\epsilon-C_{1}}{4q\epsilon},\frac{3}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)^{2}}{\left(\frac{\sqrt{2}qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{4p^{2}t^{\alpha}}{\Gamma(1+\alpha)}}\right]))$ $/(C_{2}HermiteH\left[\frac{-2q\epsilon+C_{1}}{2q\epsilon},\frac{\sqrt{q\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)}{p\sqrt{\left(\frac{\sqrt{2}qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{4q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}}}\right]+$ $Hypergeometric1F_{1}\left[\frac{2q\epsilon-C_{1}}{4q\epsilon},\frac{1}{2},\frac{\epsilon\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)^{2}}{\left(\frac{\sqrt{2}qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{4\epsilon p^{2}t^{\alpha}}{\Gamma(1+\alpha)}}\right]))),$
24	$\epsilon 3+2+4+5$	$Z=\frac{X\epsilon^{2}-T(-1+\epsilon)-\epsilon}{\epsilon^{2}\sqrt{T^{2}+2T\epsilon}}$ $W=\frac{F(Z)}{\sqrt{T^{2}+2T\epsilon}}$ $-\frac{T+2\epsilon+X\epsilon-T\epsilon}{\epsilon q^{-1}(T+2\epsilon)}$	$u(x,t)=\frac{1}{\frac{q\epsilon^{2}t^{\alpha}}{\Gamma(1+\alpha)}+2\epsilon^{3}}\times(((\frac{-q^{2}t^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-2q\epsilon$ $-2q\epsilon^{2}-\frac{pq\epsilon^{2}x^{\beta}}{\Gamma(1+\beta)}+\frac{q^{2}t^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-q\epsilon))$ $+\frac{1}{\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}}}\times$ $((2p^{2}(C_{2}(-\frac{q\left(\frac{p\epsilon x^{\beta}}{\Gamma(1+\beta)}-\frac{\frac{qt^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-\epsilon}{\epsilon}\right)}{\sqrt{\left(\frac{qp^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{2q\epsilon p^{4}t^{\alpha}}{\Gamma(1+\alpha)}}}$ $\times HermiteH\left[\frac{-2q\epsilon+C_{1}}{2q\epsilon},\frac{\sqrt{q\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{\frac{qt^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-\epsilon}{\epsilon^{2}}\right)}{p\sqrt{\left(\frac{\sqrt{2}qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{4q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}}}\right]$ $+\frac{(-2q\epsilon+C_{1})}{p\sqrt{2q\epsilon}}\times$ $HermiteH\left[\frac{-4q\epsilon+C_{1}}{2q\epsilon},\frac{\sqrt{q\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{\frac{qt^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-\epsilon}{\epsilon^{2}}\right)}{\sqrt{\left(\frac{\sqrt{2}pqt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{4q\epsilon\sqrt{p}t^{\alpha}}{\Gamma(1+\alpha)}}}\right])$

Table 15. Invariant solution of subalgebras $3+4,2,\ \epsilon 1+2+4+5,2+4,\ \epsilon 3+2+5,2+\mu 4$ and $\epsilon 3+4+5,2+\mu 4$

No	subalgebra	invariant	solution for Eq.(3.1)
			$-\frac{q\epsilon\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{\frac{qt^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-\epsilon}{\epsilon^{2}}\right)}{\sqrt{\left(\frac{qp^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{2q\epsilon p^{4}t^{\alpha}}{\Gamma(1+\alpha)}}}\times Hypergeometric1F_{1}\left[\frac{2q\epsilon-C_{1}}{4q\epsilon},\frac{1}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{\frac{qt^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-\epsilon}{\epsilon^{2}}\right)^{2}}{\left(\frac{\sqrt{2}pt^{\alpha}}{q\sqrt{\epsilon}\Gamma(1+\alpha)}\right)^{2}+\frac{4p^{2}t^{\alpha}}{\Gamma(1+\alpha)}}\right]$ $+\frac{(2q\epsilon-C_{1})\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{qt^{\alpha}}{\epsilon\Gamma(1+\alpha)}\right)}{2p^{2}\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\left(\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}\right)}}\times Hypergeometric1F_{1}\left[\frac{6q\epsilon-C_{1}}{4q\epsilon},\frac{3}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{\frac{qt^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-\epsilon}{\epsilon^{2}}\right)^{2}}{\left(\frac{\sqrt{2}pt^{\alpha}}{q\sqrt{\epsilon}\Gamma(1+\alpha)}\right)^{2}+\frac{4p^{2}t^{\alpha}}{\Gamma(1+\alpha)}}\right]))$ $/(C_{2}HermiteH\left[\frac{-2q\epsilon+C_{1}}{2q\epsilon},\frac{\sqrt{q\epsilon}\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{\frac{qt^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-\epsilon}{\epsilon^{2}}\right)}{\sqrt{2}p\sqrt{\left(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\frac{2q\epsilon t^{\alpha}}{\Gamma(1+\alpha)}}}\right]+$ $Hypergeometric1F_{1}\left[\frac{2q\epsilon-C_{1}}{4q\epsilon},\frac{1}{2},\frac{\left(\frac{px^{\beta}}{\Gamma(1+\beta)}-\frac{\frac{qt^{\alpha}(-1+\epsilon)}{\Gamma(1+\alpha)}-\epsilon}{\epsilon^{2}}\right)^{2}}{\left(\frac{\sqrt{2}pt^{\alpha}}{q\sqrt{\epsilon}\Gamma(1+\alpha)}\right)^{2}+\frac{4p^{2}t^{\alpha}}{\Gamma(1+\alpha)}}\right])))$,
25	$3+4,2$	$Z=T$ $W=\frac{F(Z)-\sqrt{T}q}{\sqrt{T}}$	$u(x,t)=C_{1}-q$
26	$\epsilon 1+2+4+5,2+4$	$Z=T$ $W=\frac{F(Z)}{\sqrt{T^{2}+\epsilon}}-\frac{q(TZ\epsilon+\sqrt{T^{2}+\epsilon})}{\epsilon\sqrt{T^{2}+\epsilon}}$	$u(x,t)=C_{1}-\frac{q}{\epsilon}$
28	$\epsilon 3+2+5,2+\mu 4$	$Z=T$ $W=\frac{F(Z)}{\sqrt{T^{2}+2T\epsilon}}-\frac{qZ\sqrt{T^{2}+2T\epsilon}}{(T+2\epsilon)}+\frac{q}{\epsilon^{2}}$	$u(x,t)=\frac{q}{\epsilon^{2}}-\frac{\frac{q^{2}t^{\alpha}}{\Gamma(1+\alpha)}\sqrt{(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{2}+\frac{2qt^{\alpha}\epsilon}{\Gamma(1+\alpha)}}}{(T+2\epsilon)}$ $+\frac{1}{\sqrt{(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{2}+\frac{2qt^{\alpha}\epsilon}{\Gamma(1+\alpha)}}}\times(\sqrt{(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{2}+\frac{2qt^{\alpha}\epsilon}{\Gamma(1+\alpha)}}C_{1}$ $+(q(-\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}+2\epsilon(4(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{6}+12\epsilon(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{5}$ $+6\epsilon^{2}(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{4}+3\epsilon^{3}(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{3}+74\epsilon^{4}(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{2}$ $+\frac{120\epsilon^{5}qt^{\alpha}}{\Gamma(1+\alpha)})+60\epsilon^{4}\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+2\epsilon)^{2}$ $\ln\left[\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}+\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+2\epsilon}\right]))/(3\epsilon^{3}(\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+2\epsilon)^{\frac{3}{2}}))$
29	$\epsilon 3+4+5,2+\mu 4$	$Z=T$ $W=\frac{F(Z)}{\sqrt{T^{2}+2T\epsilon}}-\frac{q(X+2)}{(T+2\epsilon)}$	$u(x,t)=\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}}+\sqrt{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+2\epsilon}$

Table 16. Invariant solution of subalgebras $\epsilon 3+5,2+\mu 4$ and $\epsilon 3+2+4+5,2+\mu 4$

No	subalgebra	invariant	solution for Eq.(3.1)
30	$\epsilon 3+5,2+\mu 4$	$Z=T$ $W=\frac{F(Z)-qTZ}{\sqrt{T^{2}+2T\epsilon}}$	$u(x,t)=C_{1}$
31	$\epsilon 3+2+4+5,2+\mu 4$	$Z=T$ $W=\frac{q(1-\epsilon)}{\epsilon^{2}}+\frac{F(Z)}{\sqrt{T^{2}+2T\epsilon}}-\frac{qZ\sqrt{T^{2}+2T\epsilon}}{(T+2\epsilon)}$	$u(x,t)=\frac{q(1-\epsilon)}{\epsilon^{2}}-\frac{\frac{q^{2}t^{\alpha}}{\Gamma(1+\alpha)}\sqrt{(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{2}+\frac{2qt^{\alpha}\epsilon}{\Gamma(1+\alpha)}}}{\frac{qt^{\alpha}}{\Gamma(1+\alpha)}+2\epsilon}$ $+\frac{1}{\sqrt{(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{2}+\frac{2qt^{\alpha}\epsilon}{\Gamma(1+\alpha)}}}\times$ $(q(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{2}+\sqrt{(\frac{qt^{\alpha}}{\Gamma(1+\alpha)})^{2}+\frac{2qt^{\alpha}\epsilon}{\Gamma(1+\alpha)}}C_{1})$

Notes:
The subalgebra $\{1,5\}$ have the same solution as the subalgebra $\{5\}$.
Also, the subalgebra
$\{3+2,2\},\{\epsilon 1+3+4,\delta 2+4\},\{\epsilon 1+2+3,\delta 2+4\},\{\epsilon 1+3,\delta 2+4\},\{\epsilon 1+2+3+4,\delta 2+4\},$

$\{\epsilon 1 + 3 + 4, \delta 2 + 4\}$ have the same solution as the subalgebra $\{3 + 4, 2\}$.
The subalgebras $\{3, 5\}$, $\{5 - \gamma 1, 3 + 1\}$, $\{3 + 5, 1 + \gamma 3\}$, $\{1 - \gamma 5, 3 + 5\}$, $\{3 + 4, 4\}$, $\{3 + 2, 4\}$, $\{\epsilon 1 + 4 + 5, 2 + 4\}$, $\{\epsilon 1 + 2 + 5, 2 + 4\}$, $\{\epsilon 1 + 5, 2 + 4\}$ are not invariant.

7 Conclusion

In this article, An attempt was made to clarify the application of optimal system of subalgebras method to study time fractional generalized Burgers equation (3.1). The main goal of our work has been to obtain exact solutions of time fractional generalized Burgers equation. This goal has been achieved by using optimal system of subalgebras method.

The time fractional generalized Burgers equation with the transformations is chosen to illustrate the suggested method. As results, based on symbolic computation system Mathematica program, exact solutions of the time fractional generalized Burgers equation are obtained.

Disclaimer

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Competing Interests

Authors have declared that no competing interests exist.

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