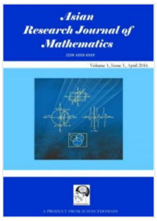

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Categorization of n-inner Product Space

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Authors' contributions

This work was carried out in collaboration between the two authors. The authors read and approved the final manuscript.

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Abstract

This paper is dealt with some properties of an n-inner product space with $n \geq 2$.The motivation of this paper is to establish the explicit forms of n-inner product space via an n-normed linear space. Some inter related results among n-normed linear space and n-inner product space are also shown here.

Keywords: n-inner product space; n-normed linear space.

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1 Introduction

In 1928, Menger [1] published the proof of a beautiful characterization of those metric spaces that are isometrically embeddable in n-dimensional Euclidean space E^n. In 1963,Gähler [2] published first one of his several research article entitled “2-metric spaces and their topological structure”, dealing with spaces on which is defined what we call a 2-metric. The second article written by Gähler [3] over 2-normed linear spaces is limited to study the special class of 2-metric spaces which are linear on which 2-norm is defined. In continuation of investigation on the topological property of such spaces, Gähler [4] had been succeeded to prove that 2-normed linear space are normable and uniformable provided the dimension of the space greater than one. Moreover, Gähler had also been able to show that if the space is 2-inner product space then it is possible to define a 2-norm on it,however the reverse implication is not true in general. Since 1963 many reserchers [5, 6] had studied extensively the geometric structure of a 2-metric space and a 2- normed linear space.Also in search of further scrutiny in these direction we see that the concept of a 2-inner product and 2-inner product spaces coincide with of the concept of natural inner product and inner product space. White et al. [7] and Diminnie et al. [8, 9] introduced the concept of a 2-inner product space and showed some characterization on it. In resent past the concept of 2-norm and concept of 2-inner product was further extended to an n-norm and an n-inner product and obtained some analogue properties of a normed linear space and an inner product space on it.

Motivated by the background of these literatures we have been able to prove some properties of an n-inner product space together with the characterization of its completeness property via n-norm.

2 Preliminaries

Definition 2.1. [10] Let n be a positive integer and X be a linear space of dimension greater than or equal to n. A real valued function $\langle .,.|...,.\rangle$ is defined on $\underbrace{X \times X \times .... \times X} = \mathbf{X}^{n+1}$ satisfying the following conditions

(IP1) $\langle x_1, x_1|x_2, ..., x_n\rangle \geq 0$ for any $x_1, x_2, ..., x_n \in X$ and $\langle x_1, x_1|x_2, ..., x_n\rangle = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent vectors,

(IP2) $\langle x_1, x_1|x_2, ..., x_n\rangle = \langle x_{i_1}, x_{i_1}|x_{i_2}, ..., x_{i_n}\rangle$ for every permutation $(i_1, i_2.....i_n)$ of $(1, 2, ...n)$,

(IP3) $\langle x, y|x_2, ..., x_n\rangle = \langle y, x|x_2, ..., x_n\rangle, \ \forall\ x, y, x_2, ..., x_n \in X$,

(IP4) $\langle \alpha x, y|x_2, ..., x_n\rangle = \alpha\langle x, y|x_2, ..., x_n\rangle\ , \forall\ x_2, ..., x_n \in X, \ \forall\ \alpha \in \mathbb{R}$,

(IP5) $\langle x + y, z|x_2, ..., x_n\rangle = \langle x, z|x_2, ..., x_n\rangle + \langle y, z|x_2, ..., x_n\rangle$, $\forall\ x, y, z, x_2, ..., x_n \in X$.

is called an n-inner product on X and the corresponding pair $(X, \langle .,.|...,.\rangle)$ is called the n-inner product space.

Example 2.2. [10] If $X = \mathbb{R}^n$ then the following function

$$\langle x, y|x_2, ..., x_n\rangle = |\det \begin{pmatrix} \langle x, y\rangle\ \langle x, x_2\rangle ....\langle x, x_n\rangle \\ \langle x_2, y\rangle\ \langle x_2, x_2\rangle ....\langle x_2, x_n\rangle \\ .... \\ .... \\ \langle x_n, y\rangle\ \langle x_n, x_2\rangle ....\langle x_n, x_n\rangle \end{pmatrix} |$$

where $x, y, x_2, ...x_n \in X$, defines an n-inner product, called the standard or (simple) n-inner product on X

Some basic properties of n-inner product$(X, \langle .,.|...,.\rangle)$ are as follows [11],[10],[12],[13].

(NIP1) $\forall\ x, y, x_2, ..., x_n \in X$,we have
$|\ \langle x, y \mid x_2, ..., x_n\rangle\ | \leq \sqrt{\langle x, x \mid x_2, ..., x_n\rangle}\sqrt{\langle y, y \mid x_2, ..., x_n\rangle}$,

(NIP2) $\forall\ x, y, x_2, ..., x_n \in X,\ \ \langle x, y \mid y, x_2, ..., x_n\rangle = 0$

(NIP3) $\forall\ x, y, x_2, ..., x_n \in X$ and $\forall\ \alpha \in \mathbb{R}$, $\langle x, y|\alpha x_2, ..., x_n\rangle = \alpha^2\langle x, y|x_2, ..., x_n\rangle$

The first inequality (NP1) is known as extension of Cauchy-Buniakowski's inequality.

(NIP4) $\forall\ x, y, z, w, x_2, ..., x_n \in X$,we have

$$\begin{aligned}\langle x, y|z + w, x_2, ..., x_n\rangle &= \langle x, y|z, x_2, ..., x_n\rangle + \langle x, y|w, x_2, ..., x_n\rangle \\ &\quad + \frac{1}{2}[\langle z, w|x + y, x_2, ..., x_n\rangle - \langle z, w|x - y, x_2, ..., x_n\rangle]\end{aligned}$$

.

Definition 2.3. [14] Under the same assumption on X, let $(\|., ..., .\|)$ be non negative real valued function from $\underbrace{X \times X \times .... \times X} = \mathbf{X}^n :\to \mathbb{R}$ satisfying the following conditions:

(N1) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n \in X$ are linearly dependent.

(N2) $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ...x_n \in X$.

(N3) $\|x_1, x_2, ..., \alpha x_n\| = |\alpha|\|x_1, x_2, ...x_n\|$, for every $\alpha \in \mathbb{R}, x_1, x_2, ...x_n \in X$.

(N4) $\|x_1, x_2, ..., x_{n-1}, y + z\| \leq \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$,
for all $y, z, x_1, x_2, ..., x_{n-1} \in\ X$.

Then $\|., ..., .\|$ is called an n-norm on X and the corresponding pair $(X, \|., ..., .\|)$ is called n-normed linear space.

Example 2.4. [14] The space $X = \mathbb{R}^n$ equipped with the following n-norm;

$$\|x_1, x_2, ..., x_n\|_E = |\det\begin{pmatrix} x_{11}\ x_{12}.........x_{1n} \\ x_{21}\ x_{22}.........x_{2n} \\ .... \\ .... \\ x_{n1}\ x_{n2}...........x_{nn}\end{pmatrix}|$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in})$ for each $i = 1, 2, ..., n$.

Some basic properties of an n-normed linear space $(X, \|., ..., .\|)$ are as follows [15]

(NN1) $\|x_1, x_2, ..., x_n\| \geq 0\ \forall\ x_1, x_2, ..., x_n \in X$,

(NN2) $\|x_1, x_2, ..., x_n + \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_{n-1}x_{n-1}\| = \|x_1, x_2, ..., x_n\|$
$\forall\ x_1, x_2, ..., x_n \in X,\ \forall\ \alpha_1, ...\alpha_{n-1} \in \mathbb{R}$.

In any linear n-inner product space $(X, \langle ., .|..., .\rangle)$,we define an n-norm by [14],[16]

$$\|x_1, x_2, ..., x_n\| = \sqrt{\langle x_1, x_1|x_2, ..., x_n\rangle}\ \ \forall x, y, x_2, ..., x_n \in X.$$

One can also observe the following [15],[14],[16]:

(NN3) $\|x + y, x_2, ..., x_n\|^2 + \|x - y, x_2, ..., x_n\|^2 = 2(\|x, x_2, ..., x_n\|^2 + \|y, x_2, ..., x_n\|^2)$.

(NIN1) $4\langle x, y|x_2, ..., x_n\rangle = \|x + y, x_2, ..., x_n\|^2 - \|x - y, x_2, ..., x_n\|^2$.

Equality is known as extension of parallelogram law.
On the other hand if $(X, \|., ..., .\|)$ is an n-normed linear space in which the condition $\|x+y, x_2, ...x_n\|^2 + \|x-y, x_2, ..., x_n\|^2 = 2(\|x, x_2, ..., x_n\|^2 + \|y, x_2, ..., x_n\|^2)$ is satisfied for all $x, y, z, x_2, ..., x_n \in X$ then n-inner product $(\langle ., .|..., .\rangle)$ on X is defined by

(IIN2) $\langle x, y|x_2, ..., x_n\rangle = \frac{1}{4}(\|x + y, x_2, ..., x_n\|^2 - \|x - y, x_2, ..., x_n\|^2)$.

For further detail refer to [10, 17].

3 Some Basic Lemmas

Lemma 3.1. *In n-inner product space, we have the following*

(i) $\|x+y, y+z, x_3, ..., x_n\| = \|x-z, y+z, x_3, ..., x_n\| = \|x+y, x-z, x_3, ..., x_n\|$

(ii) $\|x+y, y-z, x_3, ..., x_n\| = \|x+z, y-z, x_3, ..., x_n\| = \|x+y, x+z, x_3, ..., x_n\|$

(iii) $\|x-y, y+z, x_3, ..., x_n\| = \|x+z, y+z, x_3, ..., x_n\| = \|x-y, x+z, x_3, ..., x_n\|$

(iv) $\|x-y, y-z, x_3, ..., x_n\| = \|x-z, y-z, x_3, ..., x_n\| = \|x-y, x-z, x_3, ..., x_n\|$

Proof.

$$\begin{aligned}(i)\ \|x+y, y+z, x_3, ..., x_n\| &= \|(x+y)-(y+z), y+z, x_3, ..., x_n\|\ (By(NN2)) \\ &= \|x-z, y+z, x_3, ..., x_n\|\end{aligned}$$

Again,

$$\begin{aligned}\|x+y, y+z, x_3, ..., x_n\| &= \|x+y, (x+y)-(y+z), x_3, ..., x_n\|\ (By(NN2)) \\ &= \|x+y, x-z, x_3, ..., x_n\|\end{aligned}$$

The proofs of (ii)-(iv) are similar. □

Lemma 3.2. *In any n-inner product space X,the followings hold:*

(i) $\|x+y, y+z, x_3, ..., x_n\|^2 = \sum +2\langle x, y|z, x_3, ..., x_n\rangle - 2\langle x, z|y, x_3, ..., x_n\rangle + 2\langle y, z|x, x_3, ..., x_n\rangle,$

(ii) $\|x+y, y-z, x_3, ..., x_n\|^2 = \sum +2\langle x, y|z, x_3, ..., x_n\rangle + 2\langle x, z|y, x_3, ..., x_n\rangle - 2\langle y, z|x, x_3, ..., x_n\rangle,$

(iii) $\|x-y, y+z, x_3, ..., x_n\| = \sum -2\langle x, y|z, x_3, ..., x_n\rangle - 2\langle x, z|y, x_3, ..., x_n\rangle + 2\langle y, z|x, x_3, ..., x_n\rangle,$

(iv) $\|x-y, y-z, x_3, ..., x_n\| = \sum -2\langle x, y|z, x_3, ..., x_n\rangle + 2\langle x, z|y, x_3, ..., x_n\rangle - 2\langle y, z|x, x_3, ..., x_n\rangle,$

where $\Sigma = \|x, y, x_3, ..., x_n\|^2 + \|x, z, x_3, ..., x_n\|^2 + \|y, z, x_3, ..., x_n\|^2$.

Proof.

$$\begin{aligned}
(i)\|x+y,y+z,x_3,...,x_n\|^2 &= \langle x+y,x+y|y+z,x_3,...,x_n\rangle \\
&= \langle x,x+y|y+z,x_3,...,x_n\rangle+\langle y,x+y|y+z,x_3,...,x_n\rangle \\
&= \langle x,x|y+z,x_3,...,x_n\rangle+\langle x,y|y+z,x_3,...,x_n\rangle \\
&\quad +\langle y,x|y+z,x_3,...,x_n\rangle+\langle y,y|y+z,x_3,...,x_n\rangle \\
&= \langle y+z,y+z|x,x_3,...,x_n\rangle+\langle y+z,y+z|y,x_3,...,x_n\rangle \\
&\quad +2\langle x,y|y+z,x_3,...,x_n\rangle \\
&= \langle y,y+z|x,x_3,...,x_n\rangle+\langle z,y+z|x,x_3,...,x_n\rangle \\
&\quad +\langle y,y+z|y,x_3,...,x_n\rangle+\langle z,y+z|y,x_3,...,x_n\rangle \\
&\quad +2\langle x,y|y+z,x_3,...,x_n\rangle \\
&= \langle y,y|x,x_3,...,x_n\rangle+\langle y,z|x,x_3,...,x_n\rangle+\langle z,y|x,x_3,...,x_n\rangle \\
&\quad +\langle z,z|x,x_3,...,x_n\rangle+\langle y,y|y,x_3,...,x_n\rangle+\langle y,z|y,x_3,...,x_n\rangle \\
&\quad +\langle z,y|y,x_3,...,x_n\rangle+\langle z,z|y,x_3,...,x_n\rangle+2\langle x,y|y+z,x_3,...,x_n\rangle \\
&= \|y,x,x_3,...,x_n\|^2+\|y,z,x_3,...,x_n\|^2+\|z,x,x_3,...,x_n\|^2 \\
&\quad +2\langle y,z|x,x_3,...,x_n\rangle+2\langle x,y|y+z,x_3,...,x_n\rangle.
\end{aligned}$$

$$\begin{aligned}
Now, \langle x,y|y+z,x_3,...,x_n\rangle &= \langle x,y|y,x_3,...,x_n\rangle+\langle x,y|z,x_3,...,x_n\rangle \\
&\quad +\frac{1}{2}[\langle y,z|x+y,x_3,...,x_n\rangle-\langle y,z|x-y,x_3,...,x_n\rangle] \\
&= \langle x,y|z,x_3,...,x_n\rangle+\frac{1}{2}[\langle y,z|x+y,x_3,...,x_n\rangle-\langle y,z|x-y,x_3,...,x_n\rangle].
\end{aligned}$$

$$\begin{aligned}
Also, \langle y,z|x+y,x_3,...,x_n\rangle &= \langle x+y-x,z|x+y,x_3,...,x_n\rangle \\
&= \langle x+y,z|x+y,x_3,...,x_n\rangle-\langle x,z|x+y,x_3,...,x_n\rangle \\
&= -\langle x,z|x+y,x_3,...,x_n\rangle.
\end{aligned}$$

$$\begin{aligned}
\langle y,z|x-y,x_3,...,x_n\rangle &= -\langle x-y-x,z|x-y,x_3,...,x_n\rangle \\
&= -\langle x-y,z|x-y,x_3,...,x_n\rangle+\langle x,z|x-y,x_3,...,x_n\rangle \\
&= \langle x,z|x-y,x_3,...,x_n\rangle.
\end{aligned}$$

$$\begin{aligned}
\langle x,y|y+z,x_3,...,x_n\rangle &= \langle x,y|z,x_3,...,x_n\rangle-\frac{1}{2}[\langle x,z|x+y,x_3,...,x_n\rangle-\langle x,z|x-y,x_3,...,x_n\rangle] \\
&= \langle x,y|z,x_3,...,x_n\rangle-\frac{1}{2}[\langle x,z|x,x_3,...,x_n\rangle+\langle x,z|y,x_3,...,x_n\rangle \\
&\quad +\frac{1}{2}(\langle x,y|x+z,x_3,...,x_n\rangle-\langle x,y|x-z,x_3,...,x_n\rangle)] \\
&\quad +\frac{1}{2}[\langle x,z|x,x_3,...,x_n\rangle+\langle x,z|-y,x_3,...,x_n\rangle \\
&\quad +\frac{1}{2}(\langle x,-y|x+z,x_3,...,x_n\rangle-\langle x,-y|x-z,x_3,...,x_n\rangle)] \\
&= \langle x,y|z,x_3,...,x_n\rangle-\langle x,z|y,x_3,...,x_n\rangle.
\end{aligned}$$

Therefore, we have,
$\|x+y,y+z,x_3,...,x_n\|^2=\sum+2\langle x,y|z,x_3,...,x_n\rangle-2\langle x,z|y,x_3,...,x_n\rangle+\ 2\langle y,z|x,x_3,...,x_n\rangle$. Now

from Lemma 3.2 we have

$$(I) \quad 4\sum = \|x+y, y+z, x_3, ..., x_n\|^2 + \|x+y, y-z, x_3, ..., x_n\|^2 + \|x-y, y+z, x_3, ...x_n\|^2 + \|x-y, y-z, x_3, ..., x_n\|^2.$$

$$(II) \quad 8\langle x, y|z, x_3, ..., x_n\rangle = [\|x+y, y+z, x_3, ..., x_n\|^2 + \|x+y, y-z, x_3, ..., x_n\|^2] - [\|x-y, y+z, x_2, ..., x_n\|^2 + \|x-y, y-z, x_3, ..., x_n\|^2].$$

□

4 Main Results

Theorem 4.1. *An n-normed linear space X is an n-inner product space if and only if (I) is true and n-inner product is given by (II).*

Proof. Suppose X is an n-inner product space. Then by lemma 3.2 (I) follows .
Assume (I) is true in an n-normed linear space X.Using (I) we have

$$\begin{aligned}(A): \quad & 4[\|z+y, x, x_3, ..., x_n\|^2 + \|x, z-y, x_3, ..., x_n\|^2 + \|z+y, z-y, x_3, ..., x_n\|^2] \\ & = \|x+y+z, 2z, x_3, ..., x_n\|^2 + \|x+y+z, 2y, x_3, ..., x_n\|^2 \\ & +\|x-y-z, 2z, x_3, ..., x_n\|^2 + \|x-y-z, 2y, x_3, ..., x_n\|^2 \\ & = 4[\|x+y+z, z, x_3, ..., x_n\|^2 + \|x+y+z, y, x_3, ..., x_n\|^2 \\ & +\|x-y-z, z, x_3, ..., x_n\|^2 + \|x-y-z, y, x_3, ..., x_n\|^2] \\ & = 4[\|z, x+y, x_3, ..., x_n\|^2 + \|y, x+z, x_3, ..., x_n\|^2 + \|z, x-y, x_3, ..., x_n\|^2 \\ & +\|y, x-z, x_3, ..., x_n\|^2].\end{aligned}$$

$$\begin{aligned}(B): \quad & 4[\|z+x, y, x_3, ..., x_n\|^2 + \|z-x, y, x_3, ..., x_n\|^2 + \|z+x, z-x, x_3, ..., x_n\|^2] \\ & = \|x+y+z, 2z, x_3, ..., x_n\|^2 + \|x+y+z, 2x, x_3, ..., x_n\|^2 \\ & +\|z+x-y, 2z, x_3, ..., x_n\|^2 + \|z+x-y, 2x, x_3, ..., x_n\|^2 \\ & = 4[\|x+y+z, z, x_3, ..., x_n\|^2 + \|x+y+z, x, x_3, ..., x_n\|^2 \\ & +\|z+x-y, z, x_3, ..., x_n\|^2 + \|z+x-y, x, x_3, ..., x_n\|^2] \\ & = 4[\|z, y+x, x_3, ..., x_n\|^2 + \|x, y+z, x_3, ..., x_n\|^2 + \|z, y-x, x_3, ..., x_n\|^2 \\ & +\|x, y-z, x_3, ..., x_n\|^2].\end{aligned}$$

Adding (A)and(B), we have

$$\|x+y, z, x_3, ..., x_n\|^2 + \|x-y, z, x_3, ..., x_n\|^2 = 2[\|x, z, x_3, ..., x_n\|^2 + \|y, z, x_3, ..., x_n\|^2].$$

Therefore we have an n-inner product space with

$$4\langle x, y|z, x_3, ..., x_n\rangle = \frac{1}{4}[\|x+y, z, x_3, ..., x_n\|^2 - \|x-y, z, x_3, ..., x_n\|^2].$$

Once again using (I) we have,

$$\begin{aligned}(C): \quad & 4[\|x+y, y+z, x_3, ..., x_n\|^2 + \|x+y, y-z, x_3, ..., x_n\|^2 + \|y+z, y-z, x_3, ..., x_n\|^2\|] \\ & = \|x+2y+z, 2y, x_3, ..., x_n\|^2 + \|x+2y+z, 2z, x_3, ..., x_n\|^2 + \|x-z, 2y, x_3, ..., x_n\|^2 \\ & +\|x-z, 2z, x_3, ..., x_n\|^2. \\ & = 4[\|x+2y+z, y, x_3, ..., x_n\|^2 + \|x+2y+z, z, x_3, ..., x_n\|^2 + \|x-z, y, x_3, ..., x_n\|^2 \\ & +\|x-z, z, x_3, ..., x_n\|^2]. \\ & = 4[\|x+z, y, x_3, ..., x_n\|^2 + \|x+2y, z, x_3, ..., x_n\|^2 + \|x-z, y, x_3, ..., x_n\|^2 \\ & +\|x, z, x_3, ..., x_n\|^2].\end{aligned}$$

$$\begin{aligned}(D): \quad & 4[\|x-y,y+z,x_3,...,x_n\|^2+\|x-y,y-z,x_3,...,x_n\|^2+\|y+z,y-z,x_3,...,x_n\|^2\|] \\ & =\|x+z,2y,x_3,...,x_n\|^2+\|x+z,2z,x_3,...,x_n\|^2+\|x-2y-z,2y,x_3,...,x_n\| \\ & +\|x-2y-z,2z,x_3,...,x_n\|^2]. \\ & =4[\|x+z,y,x_3,...,x_n\|^2+\|x+z,z,x_3,...,x_n\|^2+\|x-2y-z,y,x_3,...,x_n\| \\ & +\|x-2y-z,z,x_3,...,x_n\|^2]. \\ & =4[\|x+z,y,x_3,...,x_n\|^2+\|x,z,x_3,...,x_n\|^2+\|x-z,y,x_3,...,x_n\|^2 \\ & +\|x-2y,z,x_3,...,x_n\|^2.\end{aligned}$$

Subtracting (D) from (C) and using (II) we get,

$$\begin{aligned}\langle x,y|z,x_3,...,x_n\rangle &= \frac{1}{8}[\|x+2y,z,x_3,...,x_n\|^2-\|x-2y,z,x_3,...,x_n\|^2] \\ &= \frac{1}{2}\langle x,2y|z,x_3,...,x_n\rangle \\ &= \langle x,y|z,x_3,...,x_n\rangle.\end{aligned}$$

This completes the proof. □

Theorem 4.2. *An n-normed linear space $(X,\|.,...,.\|)$ is an n-inner product space if and only if $\forall x,y,x_3,...,x_n\in X$. $N(s,t)=\|sx+y,y+tz,x_3,...,x_n\|^2$ is a function of $s^2t^2,s^2t,st^2,s^2,t^2,st$ where $s,t\in\mathbb{R}$.*

Proof. Assume that the n-normed linear space $(X,\|.,...,.\|)$ be an n-inner product space. Now $\forall x,y,z,x_3,...,x_n\in X$ and $s,t\in\mathbb{R}$,

$$\begin{aligned}N(s,t) &= \|sx+y,y+tz,x_3,...,x_n\|^2 \\ &= \langle sx+y,sx+y|y+zt,x_3,...,x_n\rangle \\ &= \langle sx,sx+y|y+zt,x_3,...,x_n\rangle+\langle y,sx+y|y+zt,x_3,...,x_n\rangle \\ &= \langle sx,sx|y+zt,x_3,...,x_n\rangle+\langle y,sx|y+zt,x_3,...,x_n\rangle+\langle sx,y|y+zt,x_3,...,x_n\rangle \\ &\quad +\langle y,y|y+zt,x_3,...,x_n\rangle \\ &= \langle y+tz,y+tz|sx,x_3,...,x_n\rangle+2s\langle x,y|y+zt,x_3,...,x_n\rangle+\langle y+zt,y+zt|y,x_3,...,x_n\rangle \\ &= \langle y,y+tz|sx,x_3,...,x_n\rangle+\langle tz,y+zt|sx,x_3,...,x_n\rangle+2s\langle x,y|y+zt,x_3,...,x_n\rangle \\ &\quad +\langle y,y+tz|y,x_3,...,x_n\rangle+\langle tz,y+tz|y,x_3,...,x_n\rangle \\ &= \langle y,y|sx,x_3,...,x_n\rangle++\langle tz,y|sx,x_3,...,x_n\rangle+\langle y,tz|sx,x_3,...,x_n\rangle \\ &\quad +\langle tz,tz|sx,x_3,...,x_n\rangle+2s\langle x,y|y+tz,x_3,...,x_n\rangle+\langle y,y|y,x_3,...,x_n\rangle \\ &\quad +\langle tz,y|y,x_3,...,x_n\rangle+\langle y,tz|y,x_3,...,x_n\rangle+\langle tz,tz|y,x_3,...,x_n\rangle \\ &= s^2\langle y,y|x,x_3,...,x_n\rangle+s^2t\langle z,y|x,x_3,...,x_n\rangle+s^2t\langle y,z|x,x_3,...,x_n\rangle \\ &\quad +s^2t^2\langle z,z|x,x_3,...,x_n\rangle+2s\langle x,y|y+tz,x_3,...,x_n\rangle+t^2\langle z,z|y,x_3,...,x_n\rangle \\ &= s^2\|x,y,x_3,...,x_n\|^2+2s^2t\langle z,y|x,x_3,...,x_n\rangle+s^2t^2\|x,z,x_3,...,x_n\|^2 \\ &\quad +2s\langle x,y|y+tz,x_3,...,x_n\rangle+t^2\|y,z,x_3,...,x_n\|^2.\end{aligned}$$

From (II),it follows that

$$\begin{aligned}\langle x,y|y+tz,x_3,...,x_n\rangle &= \frac{1}{8}[(\|x+y,2y+tz,x_3,...,x_n\|^2+\|x+y,tz,x_3,...,x_n\|^2) \\ &\quad -(\|x-y,2y+tz,x_3,...,x_n\|^2+\|x-y,tz,x_3,...,x_n\|^2)].\end{aligned}$$

From lemma 3.2, we have

$$\begin{aligned}
\|x+y,2y+tz,x_3,...,x_n\|^2 &= 4\|x+y,y+\frac{t}{2}z,x_3,...,x_n\|^2\\
&= 4[\|x,y,x_3,...,x_n\|^2+\|y+\frac{t}{2}z,x_3,...,x_n\|^2\\
&\quad +\|x+\frac{t}{2}z,x_3,...,x_n\|^2+2\langle x,y|\frac{t}{2}z,x_3,...,x_n\rangle\\
&\quad -2\langle x,\frac{t}{2}z|y,z,x_3,...,x_n\rangle+2\langle y,\frac{t}{2}z|x,x_3,...,x_n\rangle]\\
&= 4\|x,y,x_3,...,x_n\|^2+t^2\|y,z,x_3,...,x_n\|^2+t^2\|x,z,x_3,...,x_n\|^2\\
&\quad +2t^2\langle x,y|z,x_3,...,x_n\rangle-4t\langle x,z,|y,x_3,...,x_n\rangle+4t\langle y,z|x,x_3,...,x_n\rangle.
\end{aligned}$$

$$\begin{aligned}
\|x+y,tz,x_3,...,x_n\|^2 &= \langle x+y,x+y|tz,x_3,...,x_n\rangle\\
&= \langle x,x+y|tz,x_3,...,x_n\rangle+\langle y,x+y|tz,x_3,...,x_n\rangle\\
&= \langle x,x|tz,x_3,...,x_n\rangle+\langle x,y|tz,x_3,...,x_n\rangle\\
&\quad +\langle y,x|tz,x_3,...,x_n\rangle+\langle y,y|tz,x_3,...x_n\rangle\\
&= t^2\|x,z,x_3,...,x_n\|^2+2t^2\langle x,y|z,x_3,...,x_n\rangle\\
&\quad +t^2\|y,z,x_3,...,x_n\|^2.
\end{aligned}$$

$$\begin{aligned}
\|x-y,2y+tz,x_3,...,x_n\|^2 &= 4\|x-y,y+\frac{t}{2}z,x_3,...,x_n\|\\
&= 4[\|x,y,x_3,...,x_n\|^2+\|y,\frac{t}{2}z,x_3,...,x_n\|^2\\
&\quad +\|x,\frac{t}{2}z,x_3,...,x_n\|^2-2\langle x,y|\frac{t}{2}z,x_3,...,x_n\rangle\\
&\quad -2\langle x,\frac{t}{2}z|y,x_3,...,x_n\rangle+2\langle y,\frac{t}{2}z|x,x_3,...,x_n\rangle]\\
&= 4\|x,y,x_3,...,x_n\|^2+t^2\|y,z,x_3,...,x_n\|^2+t^2\|x,z,x_3,...,x_n\|^2\\
&\quad -2t^2\langle x,y|z,x_3,...,x_n\rangle+4t\langle x,z|y,x_3,...,x_n\rangle+4t\langle y,z|x,x_3,...,x_n\rangle
\end{aligned}$$

and $\|x-y,tz,x_3,...,x_n\|^2=t^2\|x,z,x_3,...x_n\|^2-2t^2\langle x,y|z,x_3,...x_n\rangle+t^2\|y,z,x_3,...x_n\|^2$. Thus,

$$\langle x,y|y+tz,x_3,...,x_n\rangle=t^2\langle x,y|z,x_3,...,x_n\rangle-t\langle x,z|y,x_3,...,x_n\rangle.$$

$$\begin{aligned}
So, N(s,t) &= s^2t^2\|x,z,x_3,...,x_n\|^2+2s^2t\langle y,z|x,x_3,...,x_n\rangle+2st^2\langle x,y|z,x_3,...,x_n\rangle\\
&\quad +s^2\|x,y,x_3,...,x_n\|+t^2\|y,z,x_3,...,x_n\|^2-2st\langle x,z|y,x_3,...,x_n\rangle.
\end{aligned}$$

Therefore, $N(s,t)=\|sx+y,y+tz,x_3,...,x_n\|^2$ is a function of $s^2t^2, s^2t, st^2, s^2, t^2, st$, where $s,t\in\mathbb{R}$.
Conversely,Let $N(s,t)=\|sx+y,y+tz,x_3,...,x_n\|^2=as^2t^2+bs^2t+cs^2+dst^2+et^2+fst$ be a function of $s^2t^2, s^2t, st^2, s^2, t^2, st$ where $s,t\in\mathbb{R}$.
We have, the following
$N(1,1)=\|x+y,y+z,x_3,...,x_n\|^2$.
$N(1,-1)=\|x+y,y-z,x_3,...,x_n\|^2$.
$N(-1,1)=\|x-y,y+z,x_3,...,x_n\|^2$.
$N(-1,-1)=\|x-y,y-z,x_3,...,x_n\|^2$.
Therefore,

$$\begin{aligned}
N(1,1) &+ N(1,-1)+N(-1,1)+N(-1,-1)\\
&= \|x+y,y+z,x_3,...,x_n\|^2+\|x+y,y-z,x_3,...,x_n\|^2\\
&+ \|x-y,y+z,x_3,...,x_n\|^2+\|x-y,y-z,x_3,...,x_n\|^2\\
&= 4(a+c+e).
\end{aligned}$$

Now we have the following,
$N(1,0) = \|x+y, y, x_3, ..., x_n\|^2 = \|x, y, x_3, ..., x_n\|^2 = c.$
$N(0,1) = \|y, y+z, x_3, ..., x_n\|^2 = \|y, z, x_3, ..., x_n\|^2 = e.$
Therefore we have,

$$\begin{aligned}\frac{\|sx+y, y+tz, x_3, ..., x_n\|^2}{s^2t^2} &= \|x+\frac{y}{s}, \frac{y}{t}+z, x_3, ..., x_n\|^2 \\ &= a+\frac{b}{t}+\frac{c}{t^2}+\frac{d}{s}+\frac{e}{s^2}+\frac{f}{st}.\end{aligned}$$

$$\begin{aligned}So, \|x, z, x_3, ..., x_n\|^2 &= \lim_{s,t\to\infty}\|x+\frac{y}{s}, \frac{y}{t}+z, x_3, ...x_n\| \\ &= \lim_{s,t\to\infty}\frac{\|sx+y, y+tz, x_3, ..., x_n\|^2}{s^2t^2} \\ &= \lim_{s,t\to\infty}(a+\frac{b}{t}+\frac{c}{t^2}+\frac{d}{s}+\frac{e}{s^2}+\frac{f}{st}) \\ &= a.\end{aligned}$$

Hence we have,

$$\begin{aligned}4\Sigma &= 4(\|x, y, x_3, ..., x_n\|^2 + \|x, z, x_3, ..., x_n\|^2 + \|y, z, x_3, ..., x_n\|^2) \\ &= \|x+y, y+z, x_3, ..., x_n\|^2 + \|x+y, y-z, x_3, ..., x_n\|^2 \\ &\quad +\|x-y, y+z, x_2, ...x_n\|^2 + \|x-y, y-z, x_3, ..., x_n\|^2.\end{aligned}$$

So by Theorem 4.1, $(X, \|., ..., .\|)$ is an n-inner product space. □

5 Some Consequences

Corollary 5.1. *An n-normed linear space* $(X, \|., ..., .\|)$ *is an n-inner product space if and only if* $\forall x, y, x_3, ..., x_n \in X. N(s,t) = \|sx+y, y-tz, x_3, ..., x_n\|^2$ *is a function of* $s^2t^2, s^2t, st^2, s^2, t^2, st$ *where* $s, t \in \mathbb{R}$.

Corollary 5.2. *An n-normed linear space* $(X, \|., ..., .\|)$ *is an n-inner product space if and only if* $\forall x, y, x_3, ..., x_n \in X. N(s,t) = \|sx-y, y+tz, x_3, ..., x_n\|^2$ *is a function of* $s^2t^2, s^2t, st^2, s^2, t^2, st$ *where* $s, t \in \mathbb{R}$.

Corollary 5.3. *An n-normed linear space* $(X, \|., ..., .\|)$ *is an n-inner product space if and only if* $\forall x, y, x_3, ..., x_n \in X. N(s,t) = \|sx-y, y-tz, x_3, ..., x_n\|^2$ *is a function of* $s^2t^2, s^2t, st^2, s^2, t^2, st$ *where* $s, t \in \mathbb{R}$.

6 Conclusion

The notion of n-norm introduce by Gähler in the generalization of concept of length,area and volume in a real vector space [2],[3]. The objects under consideration on such space are n-dimensional parallelopipeds. The idea of n-inner product could be taken into consideration when angle is measured between two n-dimensional parallelopipeds each having the same $(n-1)$ dimension.

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Competing Interests

Authors have declared that no competing interests exist.

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