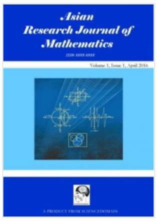

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A New and Simple Matrix Inversion Method Using Dodgson's Condensation

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

This article furnishes a new and simple matrix inversion method which makes full use of the condensation technique of the author of *Alice's Adventures in Wonderland*, Charles Dodgson. A special feature of this article is the adoption of Bhaskara's Law of Impending Operation on Zero in overcoming the problem of division by zero whenever zero appears as a divisor in the condensation technique of Dodgson.

Keywords: Matrix; determinant; division by zero; Bhaskara's law of impending operation on zero; inverse of matrices; Dodgson Condensation; Cofactor matrix.

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1 Introduction

Given any matrix, one knows, by the standard method of finding the minor of each element [1], how to compute the cofactor matrix containing the minors with their prescribed signs, and hence

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the inverse of the original matrix, obtained by dividing the transpose of the cofactor matrix by the determinant of the original matrix [2], [3], [4], [5], [6]. This method, practicable only for the second and third orders, becomes tedious and painful when it is adopted in computing the inverses of higher–order matrices [7], [8], [9], [10], [11].

Another method, more efficient, of computing the inverses of matrices is that due to Jordan, often called Gauss–Jordan method [7], [12], [6], [8]. This method involves setting up the $n \times 2n$ matrix $\begin{bmatrix} \mathbf{D} & \mathbf{I} \end{bmatrix}$ and applying elementary row operations to this matrix to convert the left half to the identity matrix $\mathbf{I}$. Clearly, in doing this, the right half will be converted to a matrix; that is, the inverse matrix $\mathbf{D^{-1}}$ will automatically be constructed in the right half as the left half is converted to the identity [13], [14], [15]. However, the method has some weakness. Even though the original matrix has integer entries, fractions are introduced along the way, making the method somewhat inefficient by requiring additional time and space to calculate the inverse.

The determination of the inverses of matrices is of great concern to mathematicians, engineers and the scientists [12]. Muen Han and Zhaolin Jiang consider the inverse of a skew Poeplitz matrix and a skew Peankel matrix involving Perrin numbers [16]. Ivan I. Kyrchei discusses Cramer's rule for generalized inverse solutions [5]. Karawia furnishes an algorithm for computing the inverse of any nonsingular heptadiagonal matrix [17]. Jinyu Chen gives explicit formulae for the inverses of symmetric Poeplitz and Qoeplitz matrices by constructing the transformation matrices [18].

The cardinal aim of this paper is to introduce a novel and fraction–free method of computing the inverses of any kind of matrix without wasting time and space. This approach makes use of the well-known condensation method of the writer of the famous *Alice's Adventures in Wonderland*, Charles Dodgson.

The rest of this paper is structured into three sections. Because some understanding of the theory of Dodgson's condensation of determinants is required to compute the inverses of matrices to which this paper is mainly devoted, we will discuss Dodgson's condensation first, and Section 2 is set up for this purpose. Section 3 deals with Bhaskara law of impending operation on zero and its application in overcoming the problem of division by zero which sometimes arise in the use of Dodgson's condensation. Section 4 deals with the use of Dodgson's condensation in computing the inverses of matrices. It is assumed that the reader is familiar with the elementary theorems of matrices and determinants.

2 Dodgson's Condensation

Charles L. Dodgson, also known as Lewis Carroll, made significant mathematical discoveries, one of which is his condensation method which has had the greatest influence on subsequent mathematical discoveries. He described his method this way: for any n by n block, i.e. matrix [19], [20]

> Compute the determinant of every 2 by 2 minor consisting of four adjacent terms. These values become the terms of a second block having $n-1$ rows and $n-1$ columns. [In modern usage, a minor is a determinant of a square sub-matrix formed from consecutive rows and columns.]
>
> Condense the second block in the same way, dividing each term, when found, by the corresponding term in the interior of the first block (the block that remains when the first and last rows and columns are erased).
>
> Repeat this process until the block is condensed to a single term, which will be the required value. Note that in condensing any block r in the series, the terms found must be divided by the corresponding terms in the interior of the $r-1$th block.

To make the method clear, we consider the matrix :

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 4 \\ 3 & 5 & -1 \end{bmatrix}. \tag{2.1}$$

We compute its determinant by condensing it to

$$\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 5 & -1 \end{vmatrix} \end{bmatrix}$$

which when evaluated gives

$$\begin{bmatrix} -5 & 14 \\ 7 & -21 \end{bmatrix}.$$

This in turn is condensed to give the value, 7. Dividing this value by the interior, 1, of the original 3rd order matrix (2.1), we get 7 which is the value of the determinant of our original 3rd order matrix.

Again, we want to compute the determinant of the 4th order matrix

$$\begin{bmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{bmatrix} \tag{2.2}$$

using Dodgson's condensation technique. This is condensed into

$$\begin{bmatrix} \begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 0 & -4 \\ 5 & 1 \end{vmatrix} & \begin{vmatrix} -4 & 6 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 6 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ -3 & 8 \end{vmatrix} & \begin{vmatrix} 2 & 6 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 6 & -1 \\ 9 & 1 \end{vmatrix} \end{bmatrix}$$

which, when evaluated, gives

$$\begin{bmatrix} 10 & 20 & -6 \\ 8 & 28 & -1 \\ 6 & -30 & 15 \end{bmatrix}. \tag{2.3}$$

This in turn is condensed into

$$\begin{bmatrix} \begin{vmatrix} 10 & 20 \\ 8 & 28 \end{vmatrix} & \begin{vmatrix} 20 & -6 \\ 28 & -1 \end{vmatrix} \\ \begin{vmatrix} 8 & 28 \\ 6 & -30 \end{vmatrix} & \begin{vmatrix} 28 & -1 \\ -30 & 15 \end{vmatrix} \end{bmatrix}$$

which, being evaluated, furnishes

$$\begin{bmatrix} 120 & 148 \\ -408 & 390 \end{bmatrix}.$$

We divide each element of the above 2×2 matrix by the corresponding element of the interior matrix of the original 4th order matrix (2.2)

$$\begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix},$$

and have

$$\begin{bmatrix} \frac{120}{5} & \frac{148}{1} \\ \frac{-408}{2} & \frac{390}{6} \end{bmatrix}$$

which gives

$$\begin{bmatrix} 24 & 148 \\ -204 & 65 \end{bmatrix}$$

which, when evaluated, gives the value of 31752. Dividing this value by the interior, 28, of the derived 3rd order matrix (2.3), we get 1134 which is the value of our original 4th order matrix.

The simplest way of presenting the workings appears to be to arrange the series of matrices one under another, as it is displayed below; it will then be found very easy to pick out the divisors (in the interior matrices):

$$\begin{bmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 20 & -6 \\ 8 & 28 & -1 \\ 6 & -30 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 24 & 148 \\ -204 & 65 \end{bmatrix}$$

$$1134.$$

Dodgson's condensation method, being interesting and excellently suited to hand–computations, is in the first place remarkable for its exceedingly great briefness, lucidity and accuracy. It is also noteworthy as it involves the evaluation of only 2nd order determinants, the elements of which are adjacent to one another [19], [21], [20].

However, it is evident that, when zeros (which Dodgson called *ciphers* in his paper [20]) appear in the interior of the original matrix or any one of the derived matrices, the process cannot be continued because of the emergence of division by zero[20]. A solution to this problem, as Dodgson suggests, is to recommence the operation by first rearranging the original matrix by transferring the top row to the bottom or the bottom row to the top so that the zero, when it occurs, is now found in an exterior row [19], [21], [22]. The merit of this solution is that "there is only *one* new row to be computed; the other rows are simply copied from the work already done"[20].

Suppose now we want to find the value of the determinant of the matrix

$$\begin{bmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{bmatrix}.$$

We compute as follows:

$$\begin{bmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -5 & -3 & -1 \\ -3 & -3 & -3 & 3 \\ 3 & 3 & 3 & -1 \\ -5 & -3 & -1 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -15 & 6 & 12 \\ 0 & 0 & 6 \\ 6 & -6 & 8 \end{bmatrix}.$$

We cannot continue the operation because of the zero which occurs in the interior of the derived 3rd order matrix. Division by zero will occur. So we rearrange the original 5th order matrix by moving the top row to the bottom and moving all the other rows up once, and recommence the operation:

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \\ 2 & -1 & 2 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 & -3 & 3 \\ 3 & 3 & 3 & -1 \\ -5 & -3 & -1 & -5 \\ 3 & -5 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 6 \\ 6 & -6 & 8 \\ -17 & 8 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 12 \\ 18 & 40 \end{bmatrix}$$

$$36.$$

There is another means of overcoming this problem of zero divisor without recommencing the condensation process. It is the use of Bhaskara's Law of Impending Operation on Zero which is discussed in the next section.

3 Bhaskara's Law of Impending Operation on Zero

The first master of the use of zero, the number representing absolute nothing [23],[24] [25] was the Indian mathematician Brahmagupta [26]. About 500 years later, the use of zero in calculation was then taken up by Bhaskara II. In his *Lilavati*, he gives us a mass of information on the arithmetic of zero. He was quite aware of the subtleties of treating zero as a number. In fact, he was cognizant of the eluding behaviour of zero as a divisor [27]. We read him [28]:

> The product of zero is naught, but it must be retained as a multiple of zero if any operations impend. Zero, having become a multiplier, should naught afterwards become a divisor, the definite quantity must be understood to be unchanged.

This is Bhaskara's law of impending operation on zero and it will lend us some light on which we expect some pregnant hints. Division and multiplication are inverse operations. That means that if we start with some finite number a and then divide by $\mathbf{0}$ where the bold–faced symbol $\mathbf{0}$ is a numerical value representing absolute nothing, then if we wish to preserve a, we have to multiply by $\mathbf{0}$:

$$\frac{a}{\mathbf{0}} \cdot \mathbf{0} = a.$$

If, however, we first multiply by $\mathbf{0}$, then to preserve the finite number a, we have to divide by $\mathbf{0}$:

$$\frac{a \cdot \mathbf{0}}{\mathbf{0}} = a.$$

Let there be no mistake or vagueness about this law of Bhaskara's. It is this law which underlies and forms the basis of zero analysis. Understanding it, everything is simple, regular, logical and non–paradoxical; ignoring it, the various operations seem disconnected and abstruse. All - even the most advanced - work in analysis is based on this simple law, and not on the cumbersome and innumerable rules of the method of limit.

Now, we use the symbol $\mathbf{0}$ for evaluation processes e.g when we say let $x = \mathbf{0}$ in any expression of x or when we set $x = c$ in the expression $x - c$ (comparison subtraction), *viz* $(c) - c = \mathbf{0}$ or when we put $x = -c$ in the expression $x + c$ *viz* $(-c) + c = \mathbf{0}$. It follows that setting $x = c$ in the expression $c - x$, which is also $-(x - c)$, is $-\mathbf{0}$ and putting $x = -c$ in the expression $-x - c$, which is also $-(x + c)$, is $-\mathbf{0}$. The zeros $\mathbf{0}$ and $-\mathbf{0}$ coincide at the origin of the number line and hence signify the same physical thing, absolute nothing. Still in impending operations one cannot be substituted for the other. This last statement is true also of other zeros such as $2 \cdot \mathbf{0}$, $\mathbf{0}^4$, etc. Elimination expressions (take away subtractions) such as $1 - 1$, $2 - 2$, $x - x$, $3x^2 - 3x^2$, etc which arise only during simplification of mathematical expressions or equations are considered blank or marked with the usual sign 0 [29], [30].

It may be asked: Why must we use $a \cdot \mathbf{0}$ in impending operations and not merely 0 where the usual symbol 0 represents blank or absence of quantity? The answer to this appears when we consider the following. It is well known that the product of every two numbers x and y is algebraically written as $x \cdot y$. For example, the product of 2 and 3 is $2 \cdot 3$. But, since this corresponds with a recognized finite number in the number system, that is, the number 6, we just write $2 \cdot 3 = 6$. Now, the result 6 which is the evaluation of $2 \cdot 3$ contains in itself the two factors, 2 and 3, forming it. Thus, when it is required that this number 6 be used in a future calculation where it is to be divided by any multiple of 2 or 3, the 2 or 3 contained in the number 6 will have to cancel out with the 2 or 3 in the multiple. Thus,

$$\frac{6}{10} = \frac{2 \times 3}{2 \times 5} = \frac{3}{5}$$

or

$$\frac{6}{21} = \frac{2 \times 3}{3 \times 7} = \frac{2}{7}.$$

Now, there is no number in the number system which is recognized as the result of the evaluation of the product $a \cdot \mathbf{0}$, and since this is true, it is, therefore, important to leave the product $a \cdot \mathbf{0}$ in this form if we are expecting that this result will be employed in impending operations. Thus, if we obtain $6 \cdot \mathbf{0}$ as the result from a calculation, and we are expecting that this result be used in further computations, we should not, therefore, write $6 \cdot \mathbf{0} = 0$, for the result 0 does not reveal the presence of the factor 6. But, we should use this product $6 \cdot \mathbf{0}$ in this form in the expected calculations. Thus,

For more details, the reader should read the Author's papers:

1. Bhaskara's Arithmetic Operation of Division by Zero with Application in the Foundation of the Differential Calculus, **The Journal of the Indian Mathematical Society**, Volume 84, Issue 3-4, 2017.

2. On the Operation of Division by Zero in Bhaskara's Framework: Survey, Criticisms, Modifications and Justifications, **Asian Research Journal of Mathematics** 6(2): 1-20, 2017; Article no.ARJOM.36240.

if we are anticipating that this product divides $12 \cdot \mathbf{0}$ we should work the calculation as follows:

$$\frac{12 \cdot \mathbf{0}}{6 \cdot \mathbf{0}} = 2.$$

3.1 Arithmetic Operations with Zero

It is our business now to deal with the arithmetic operations with zero, based on the fact that there is an approaching or impending operation of division by zero.

3.1.1 Zeros of First Order

The first idea that occurs on this subject is that zeros of the form $\mathbf{0}a$, also written as $\mathbf{0} \cdot a$ or $a \cdot \mathbf{0}$ and called zeros of first order, are added, subtracted, multiplied and divided just as we would any algebraic expression of the form ax where a is a constant and x is a variable. The one difference we need to keep in mind is that $a \cdot \mathbf{0}$ is absolute naught [31], [32].

We define the sum, difference, and product of zeros of first order as follows. To add these zeros, add their finite parts. If $a \cdot \mathbf{0}$ and $b \cdot \mathbf{0}$ are two zeros of the first order, then

$$a \cdot \mathbf{0} + b \cdot \mathbf{0} = (a + b) \cdot \mathbf{0}.$$

For instance,

$$3 \cdot \mathbf{0} + 4 \cdot \mathbf{0} = (3 + 4) \cdot \mathbf{0} = 7 \cdot \mathbf{0}.$$

To subtract the zeros of first order, subtract their finite parts. Thus

$$a \cdot \mathbf{0} - b \cdot \mathbf{0} = (a - b) \cdot \mathbf{0}.$$

For instance,

$$3 \cdot \mathbf{0} - 4 \cdot \mathbf{0} = (3 - 4) \cdot \mathbf{0} = -\mathbf{0}.$$

The product of a finite number a and the zero $b \cdot \mathbf{0}$ is $(a \cdot b) \cdot \mathbf{0}$. Thus, for instance,

$$2 \times 3 \cdot \mathbf{0} = 6 \cdot \mathbf{0}.$$

To multiply two zeros of first order, we multiply their finite parts and then their zero parts, observing the law of indices- $a^m \times a^n = a^{m+n}$. Thus we have

$$a \cdot \mathbf{0} \times b \cdot \mathbf{0} = (a \times b) \cdot (\mathbf{0} \times \mathbf{0}) = ab \cdot \mathbf{0}^2.$$

For example,

$$3 \cdot \mathbf{0} \times 4 \cdot \mathbf{0} = 12 \cdot \mathbf{0}^2.$$

Division of first order zeros is much like the algebraic division

$$\frac{ax}{bx} = \frac{a}{b}.$$

For the division of the zero $a \cdot \mathbf{0}$ by the zero $b \cdot \mathbf{0}$ we have

$$\frac{a \cdot \mathbf{0}}{b \cdot \mathbf{0}} = \frac{a}{b}.$$

The expression a/b is a finite number as it is the ratio of two finite numbers. Thus the division of two zeros of first order gives a finite number [31], [32].

3.1.2 Zeros of Higher Orders

We are now concerned with the operations with zeros of higher orders. A numerical zero of order n is a zero of the form $k \cdot \mathbf{0}^n$ where n is a positive integer and k is any finite number. The number k is called the finite part of the zero. The expression $\mathbf{0}^n$ is called the unit zero part of the numerical zero. The following zero is of order 5 and has a finite part of 3: $3 \cdot \mathbf{0}^5$.

The sum of two zeros of the same order is found by adding their finite parts. If $k_1 \cdot \mathbf{0}^n$ and $k_2 \cdot \mathbf{0}^n$ are two zeros of the same order n, then, using the distributive property

$$k_1 \cdot \mathbf{0}^n \pm k_2 \cdot \mathbf{0}^n = (k_1 \pm k_2) \cdot \mathbf{0}^n.$$

Consider the following example:

$$2 \cdot \mathbf{0}^3 + 3 \cdot \mathbf{0}^3.$$

This expression can be written as

$$2 \cdot \mathbf{0}^3 + 3 \cdot \mathbf{0}^3 = (2+3) \cdot \mathbf{0}^3 = 5 \cdot \mathbf{0}^3.$$

Similarly, we have the subtraction

$$2 \cdot \mathbf{0}^3 - 3 \cdot \mathbf{0}^3 = (2-3) \cdot \mathbf{0}^3 = -\mathbf{0}^3.$$

The process of addition or subtraction is performed only with the finite parts as we already mentioned. The zero part remains unchanged. Recall that $\mathbf{0}$ is the same as $1 \cdot \mathbf{0}$. The finite part is understood to be 1. Also, $-\mathbf{0}$ is the same as $-1 \cdot \mathbf{0}$.

Suppose we wish to perform the indicated addition and subtraction

$$6 \cdot \mathbf{0}^2 - 4 \cdot \mathbf{0} + 3 \cdot \mathbf{0} - 2 \cdot \mathbf{0}^2.$$

We work as follows:

$$\begin{aligned} 6 \cdot \mathbf{0}^2 - 4 \cdot \mathbf{0} + 3 \cdot \mathbf{0} - 2 \cdot \mathbf{0}^2 &= 6 \cdot \mathbf{0}^2 - 2 \cdot \mathbf{0}^2 - 4 \cdot \mathbf{0} + 3 \cdot \mathbf{0} \\ &= 4 \cdot \mathbf{0}^2 - 1 \cdot \mathbf{0} \\ &= -\mathbf{0} + 4 \cdot \mathbf{0}^2 \end{aligned}$$

Because of the commutative and associative properties, we can rearrange the expression and add zeros of like order.

If the zero $k \cdot \mathbf{0}^n$ is multiplied by the finite number c, only the finite part is involved in the multiplication. Thus, $c \times k \cdot \mathbf{0}^n = (ck) \cdot \mathbf{0}^n$. For instance, to multiply 2 by the zero $3 \cdot \mathbf{0}^5$, we only need to find the product of the finite number 2 and the finite part of $3 \cdot \mathbf{0}^5$ which is 3. So,

$$2 \times 3 \cdot \mathbf{0}^5 = 6 \cdot \mathbf{0}^5.$$

The multiplication of two numerical zeros is a third zero expressed as the product of their finite parts times the product of their unit zero parts. To multiply $3 \cdot \mathbf{0}^2$ by $2 \cdot \mathbf{0}$, we apply the commutative and associative properties of multiplication along with the properties of exponents. We then write the expression $3 \cdot \mathbf{0}^2 \times 2 \cdot \mathbf{0}$ as a product of the finite parts 3×2 times the product of the unit zero parts $\mathbf{0}^2 \times \mathbf{0}$. That is

$$3 \cdot \mathbf{0}^2 \times 2 \cdot \mathbf{0} = (3 \times 2) \cdot \big(\mathbf{0}^2 \times \mathbf{0}\big) = 6 \cdot \mathbf{0}^3.$$

Of importance in this work is the division of any power of the unit zero, say $\mathbf{0}^m$ by another power of the unit zero, $\mathbf{0}^n$. The rule is to subtract the exponent n of the zero divisor from the exponent m of the zero dividend:

$$\frac{\mathbf{0}^m}{\mathbf{0}^n} = \mathbf{0}^{m-n}.$$

Thus $\mathbf{0}, \mathbf{0}^2, \mathbf{0}^3, \mathbf{0}^4, \ldots$, divided by $\mathbf{0}$ give the respective results $1, \mathbf{0}, \mathbf{0}^2, \ldots$. The indicated division

$$\frac{8 \cdot \mathbf{0}^5}{\mathbf{0}}$$

when simplified gives $8 \cdot \mathbf{0}^4$.

Based on zeros of higher orders, we improve the principle of impending operation on zero: "If in some mathematical calculations, zeros are likely to occur frequently, then, though $a \cdot \mathbf{0}^n$ is absolute nothing, where a is a finite number and n is any positive number, one should maintain the form $a \cdot \mathbf{0}^n$ in the rest of the operations until the final operation is reached. This is because if a finite number is multiplied by zero and divided by the same zero, then the result is the finite number".

3.2 Illustrative Example given by Bhaskara

The following illustrative example given by Bhaskara will be of incalculable guild to the reader.

> A certain number is multiplied by $\mathbf{0}$ and added to half of result. If the sum so obtained is first multiplied by 3 and then divided by $\mathbf{0}$, the result is 63. Find the original number.

If the number is x, then we write

$$\frac{\left(x \cdot \mathbf{0} + \frac{1}{2} x \cdot \mathbf{0}\right) 3}{\mathbf{0}} = 63$$

which becomes

$$\frac{3x \cdot \mathbf{0} + \frac{3}{2} x \cdot \mathbf{0}}{\mathbf{0}} = 63$$

which in its turn becomes

$$\frac{6x \cdot \mathbf{0} + 3x \cdot \mathbf{0}}{2 \cdot \mathbf{0}} = 63.$$

This is simplified to

$$\frac{9x \cdot \mathbf{0}}{2 \cdot \mathbf{0}} = 63$$

which, canceling out the two zeros, becomes

$$\frac{9x}{2} = 63$$

which finally gives $x = 14$. This is the answer Bhaskara would expect.

3.3 Application in the Evaluation of Functions

Suppose we were asked to determine the value of

$$f(x) = \frac{x^2 + x - 6}{x + 3}$$

where $x = -3$. Direct substitution shows that both numerator and denominator must equal zero numbers. To obtain the expressions for these, we first factor $x^2 + x - 6$ and set $x = -3$. Thus, $x^2 + x - 6$ becomes $(x - 2)(x + 3)$ which on setting $x = -3$ becomes $(-3 - 2)(-3 + 3) = -5 \cdot \mathbf{0}$. We must retain this zero number because of the further operation of division by the denominator zero number $(-3+3) = \mathbf{0}$. Thus,

$$f(-3) = \frac{-5 \cdot \mathbf{0}}{\mathbf{0}} = -5.$$

Suppose we wish to evaluate the function

$$f(x) = \frac{\sqrt{x+1}-1}{x}$$

at $x = \mathbf{0}$. We accomplish this as follow:

$$f(\mathbf{0}) = \frac{\sqrt{\mathbf{0}+1}-1}{\mathbf{0}}.$$

To simplify this, we apply the method of rationalization, viz.

$$\begin{aligned} f(\mathbf{0}) &= \frac{\sqrt{\mathbf{0}+1}-1}{\mathbf{0}} \times \frac{\sqrt{\mathbf{0}+1}+1}{\sqrt{\mathbf{0}+1}+1} \\ &= \frac{(\sqrt{\mathbf{0}+1})^2 - 1^2}{\mathbf{0}(\sqrt{\mathbf{0}+1}+1)} \\ &= \frac{\mathbf{0}+1-1}{\mathbf{0}(\sqrt{\mathbf{0}+1}+1)} \\ &= \frac{\mathbf{0}}{\mathbf{0}(\sqrt{\mathbf{0}+1}+1)} \\ &= \frac{1}{\sqrt{\mathbf{0}+1}+1} \\ &= \frac{1}{2} \end{aligned}$$

We desire to evaluate

$$\frac{2x^7 \sin x}{2 - x^4 - 2\cos x^2}$$

at $x = \mathbf{0}$. First we evaluate the numerator at $x = \mathbf{0}$:

$$\begin{aligned} 2 \cdot \mathbf{0}^7 \sin \mathbf{0} &= 2 \cdot \mathbf{0}^7 \left(\mathbf{0} - \frac{\mathbf{0}^3}{3!} + \frac{\mathbf{0}^5}{5!} - \cdots \right) \\ &= 2 \cdot \mathbf{0}^8 \left(1 - \frac{\mathbf{0}^2}{3!} + \frac{\mathbf{0}^4}{5!} - \cdots \right) \end{aligned}$$

which is absolute nothing. Next, we evaluate the denominator at $x = \mathbf{0}$:

$$\begin{aligned} 2 - \mathbf{0}^4 - 2\cos \mathbf{0}^2 &= 2 - \mathbf{0}^4 - 2\left(1 - \frac{\mathbf{0}^4}{2!} + \frac{\mathbf{0}^8}{4!} - \frac{\mathbf{0}^{12}}{6!} + \cdots \right) \\ &= 2 - \mathbf{0}^4 - 2 + \mathbf{0}^4 - \frac{\mathbf{0}^8}{12} + \frac{\mathbf{0}^{12}}{360} - \cdots \\ &= 2 - 2 - \mathbf{0}^4 + \mathbf{0}^4 - \frac{\mathbf{0}^8}{12} + \frac{\mathbf{0}^{12}}{360} - \cdots. \end{aligned}$$

The expressions $2-2$ and $-\mathbf{0}^4 + \mathbf{0}^4$ are elimination expressions and so we consider them blank. Hence, we write

$$2 - \mathbf{0}^4 - 2\cos \mathbf{0}^2 = -\frac{\mathbf{0}^8}{12} + \frac{\mathbf{0}^{12}}{360} - \cdots = -\frac{\mathbf{0}^8}{12}\left(1 - \frac{\mathbf{0}^2}{30} + \cdots \right)$$

which is also absolute nothing. Thus the evaluation of

$$\frac{2x^7 \sin x}{2 - x^4 - 2\cos x^2}$$

at $x = \mathbf{0}$ is

$$\frac{2 \cdot \mathbf{0}^7 \sin \mathbf{0}}{2 - \mathbf{0}^4 - 2\cos \mathbf{0}^2} = \frac{2 \cdot \mathbf{0}^8 \left(1 - \frac{\mathbf{0}^2}{3!} + \frac{\mathbf{0}^4}{5!} - \cdots\right)}{-\frac{0^8}{12}\left(1 - \frac{\mathbf{0}^2}{30} + \cdots\right)}$$

which becomes

$$\frac{2 \cdot \mathbf{0}^7 \sin \mathbf{0}}{2 - \mathbf{0}^4 - 2\cos \mathbf{0}^2} = \frac{2 \cdot \mathbf{0}^8}{-\frac{\mathbf{0}^8}{12}} = -24.$$

3.4 Application in the Calculus

In the late 17th century, Leibniz and Newton sought to give a foundation for the calculus. They made use of infinitesimals and fluxions respectively, that is non-zero quantities that is less that any finite quantity but capable of being neglected or made to vanish. For instance, if we wish to find the derivative of $f(x) = x^2$, we would compute as follows:

$$f'(x) = \frac{(x+\epsilon)^2 - x^2}{\epsilon} = \frac{2x\epsilon + \epsilon^2}{\epsilon} = 2x + \epsilon.$$

It is claimed that because ϵ is infinitely small though not equal to zero, it will be convenient to regard it to be equal to zero. Many mathematicians, notably Rolle and Berkeley, railed at this because the infinitesimal and fluxional doctrines were unintelligible for them to understand.

Let us now apply Bhaskara's Law in finding the derivative of $f(x) = x^2$. This is done as follows:

$$f'(x) = \frac{(x+\mathbf{0})^2 - x^2}{\mathbf{0}} = \frac{2x \cdot \mathbf{0} + \mathbf{0}^2}{\mathbf{0}} = \frac{2x \cdot \mathbf{0}}{\mathbf{0}} + \frac{\mathbf{0}^2}{\mathbf{0}} = 2x + \mathbf{0} = 2x.$$

3.5 Application in Dodgson's Condensation of Determinants

We are led to consider now the application of Bhaskara's Law in the expansion of determinants of matrices by Dodgson's condensation. Suppose we want to find the value of the determinant of the matrix

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix}.$$

It will be seen at once that this will be impossible using Dodgson's condensation, since the interior of the matrix is 0. Division by the zero will result in an infinite value. Since there is a division by zero, it must be admitted that the calculation is to be performed in Bhaskara's framework. Thus, all the beautiful properties of zero that have been discussed so far should be ready for absolute employment. Add $\mathbf{0}$ to the blank represented by 0. This step gives rise to the matrix

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & \mathbf{0} & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

which undergoes condensation to give

$$\begin{bmatrix} 2 \cdot \mathbf{0} - 1 & 1 - 5 \cdot \mathbf{0} \\ 1 - 2 \cdot \mathbf{0} & -\mathbf{0} - 1 \end{bmatrix}.$$

This in turn is condensed to give

$$\frac{(2 \cdot \mathbf{0} - 1)(-\mathbf{0} - 1) - (1 - 5 \cdot \mathbf{0})(1 - 2 \cdot \mathbf{0})}{\mathbf{0}}$$

which becomes

$$\frac{1}{\mathbf{0}}\left[\left(-2 \cdot \mathbf{0}^2 - 2 \cdot \mathbf{0} + \mathbf{0} + 1\right) - \left(1 - 2 \cdot \mathbf{0} - 5 \cdot \mathbf{0} + 10 \cdot \mathbf{0}^2\right)\right]$$

which, being evaluated, becomes

$$\frac{1}{\mathbf{0}}\left[\left(-2 \cdot \mathbf{0}^2 - \mathbf{0} + 1\right) - \left(1 - 7 \cdot \mathbf{0} + 10 \cdot \mathbf{0}^2\right)\right].$$

This is also simplified to

$$\frac{1}{\mathbf{0}}\left(-2 \cdot \mathbf{0}^2 - \mathbf{0} + 1 - 1 + 7 \cdot \mathbf{0} - 10 \cdot \mathbf{0}^2\right)$$

which, taking $1 - 1$ as blank, is reduced to

$$\frac{1}{\mathbf{0}}\left(-12 \cdot \mathbf{0}^2 + 6 \cdot \mathbf{0}\right)$$

which, on removing the bracket, results in

$$\frac{-12 \cdot \mathbf{0}^2}{\mathbf{0}} + \frac{6 \cdot \mathbf{0}}{\mathbf{0}}.$$

This gives us

$$-12 \cdot \mathbf{0} + 6$$

which gives 6 as will be obtained if the well-known Laplace method is employed.

Let us now find the value of the determinant of the already mentioned matrix

$$\begin{bmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{bmatrix}.$$

We compute as follows:

$$\begin{bmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -5 & -3 & -1 \\ -3 & -3 & -3 & 3 \\ 3 & 3 & 3 & -1 \\ -5 & -3 & -1 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -15 & 6 & 12 \\ 0 & 0 & 6 \\ 6 & -6 & 8 \end{bmatrix}.$$

We cannot continue the operation because of the zero which occurs in the interior of the derived 3rd order matrix. To overcome this, we add $\mathbf{0}$ to any of the entries of the submatrix whose determinant is zero. Thus, we have

$$\begin{bmatrix} 5 & -5 & -3 & -1 \\ -3 & -3 & -3 & 3 \\ 3 & 3 & 3 + \mathbf{0} & -1 \\ -5 & -3 & -1 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -15 & 6 & 12 \\ \mathbf{0} & -\mathbf{0} & 6+\mathbf{0} \\ 6 & -6-\mathbf{0} & 8+\frac{5}{2}\cdot\mathbf{0} \end{bmatrix}.$$

$$\begin{bmatrix} -5\cdot\mathbf{0} & -12-10\cdot\mathbf{0} \\ 2\cdot\mathbf{0} & \dfrac{72+8\cdot\mathbf{0}-3\cdot\mathbf{0}^2}{2(3+\mathbf{0})} \end{bmatrix}.$$

The determinant of the matrix is therefore

$$\frac{1}{-\mathbf{0}}\left[-5\cdot\mathbf{0}\left(\frac{72+8\cdot\mathbf{0}-3\cdot\mathbf{0}^2}{2(3+\mathbf{0})}\right)+2\cdot\mathbf{0}(12+10\cdot\mathbf{0})\right]$$

which becomes

$$5\left(\frac{72+8\cdot\mathbf{0}-3\cdot\mathbf{0}^2}{2(3+\mathbf{0})}\right)-2(12+10\cdot\mathbf{0}).$$

Omitting **0** as it merely represents absolute nothing gives

$$5\left(\frac{72}{2(3)}\right)-2(12)$$

which is equal to 36.

4 Inverses of Matrices

In this section, I shall teach how to calculate, by means of a new method, the inverses of matrices of not only the second and third orders, but also the fourth orders. This method uses Dodgson's condensation and computes the inverse of any $n \times n$ matrix $\mathbf{D}$ as demonstrated in the following rules:

1. Form the four-quadrant matrix by putting $\mathbf{D}$ in four quadrants:

$$\begin{bmatrix} \mathbf{D} & \mathbf{D} \\ \mathbf{D} & \mathbf{D} \end{bmatrix}.$$

2. Form the interior matrix of the four-quadrant matrix by deleting the first and last rows and columns of the four-quadrant matrix.
3. Apply Dodgson's condensation by condensing the interior matrix of the four-quadrant matrix to matrix of the next lower order and continue the process until a matrix of the same order as $\mathbf{D}$ is obtained, that is an $n \times n$ matrix.

If n is odd, the final $n \times n$ matrix formed is the cofactor matrix of $\mathbf{D}$. If n is even and we give each element of the final $n \times n$ matrix formed its prescribed sign, the resulting matrix is the cofactor matrix of $\mathbf{D}$.

It is interesting to note that the determinant of $\mathbf{D}$ can be easily obtained from the above process. Apply the condensation one more time. We get an $n-1 \times n-1$ matrix. If n is odd, then the $n-1 \times n-1$ matrix formed is a matrix consisting of only the determinant of $\mathbf{D}$ as its elements. If n is even and we give each element of the $n-1 \times n-1$ matrix formed its prescribed sign, the resulting matrix consists of only the determinant of $\mathbf{D}$ as its elements.

To illustrate the ease with which this method is used to obtain the inverse of matrices, we begin with the simplest case, the 2nd order matrix

$$\mathbf{D} = \begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}.$$

By rule 1 we write the four-quadrant matrix as

$$\begin{bmatrix} 2 & 4 & 2 & 4 \\ -1 & 7 & -1 & 7 \\ 2 & 4 & 2 & 4 \\ -1 & 7 & -1 & 7 \end{bmatrix}$$

and by rule 2 we get the interior matrix of the four-quadrant matrix as

$$\begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix}.$$

We give the elements of this matrix their prescribed signs since the original matrix is of even order, and we obtain the cofactor matrix as

$$\begin{bmatrix} 7 & -(-1) \\ -4 & 2 \end{bmatrix}$$

which gives

$$\begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix}$$

which, being transposed, becomes

$$\begin{bmatrix} 7 & -4 \\ 1 & 2 \end{bmatrix}.$$

The inverse of the original matrix is, therefore,

$$\mathbf{D^{-1}} = \frac{1}{18}\begin{bmatrix} 7 & -4 \\ 1 & 2 \end{bmatrix}$$

where the value 18 is the determinant of the original matrix.

We now compute by means of this new technique the inverse of the 3×3 matrix:

$$\mathbf{D} = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}.$$

By rule 1 we write the four-quadrant matrix as

$$\begin{bmatrix} 4 & 1 & -5 & 4 & 1 & -5 \\ -2 & 3 & 1 & -2 & 3 & 1 \\ 3 & -1 & 4 & 3 & -1 & 4 \\ 4 & 1 & -5 & 4 & 1 & -5 \\ -2 & 3 & 1 & -2 & 3 & 1 \\ 3 & -1 & 4 & 3 & -1 & 4 \end{bmatrix},$$

and by applying rule 2 we get the 4×4 interior matrix of the four-quadrant matrix as

$$\begin{bmatrix} 3 & 1 & -2 & 3 \\ -1 & 4 & 3 & -1 \\ 1 & -5 & 4 & 1 \\ 3 & 1 & -2 & 3 \end{bmatrix}$$

which, by rule 3, becomes the cofactor matrix

$$\begin{bmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{bmatrix}$$

which, after famously undergoing transposition, becomes

$$\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}.$$

Here we must not give signs to the elements since the original matrix is of odd order, and we must stop condensation at this point since the derived matrix is of the same order as the original matrix **D**. We now compute the determinant of **D** as follows:

$$|\mathbf{D}| = \frac{1}{4}\begin{vmatrix} 13 & 11 \\ 1 & 31 \end{vmatrix} = 98.$$

Notice also that

$$|\mathbf{D}| = \frac{1}{3}\begin{vmatrix} 11 & -7 \\ 31 & 7 \end{vmatrix} = 98,$$

$$|\mathbf{D}| = \frac{1}{-5}\begin{vmatrix} 1 & 31 \\ 16 & 6 \end{vmatrix} = 98,$$

$$|\mathbf{D}| = \frac{1}{4}\begin{vmatrix} 31 & 7 \\ 6 & 14 \end{vmatrix} = 98.$$

Thus the inverse of **D** is

$$\mathbf{D^{-1}} = \frac{1}{98}\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}.$$

Clearly, the computation of the inverse of 2×2 and 3×3 matrices by employing Dodgson's condensation is simple and systematic and does not involve the liability of dividing by elements in the interiors of the matrices, thereby escaping the cipher problem of division by zero. This new method, therefore, deserves utmost consideration and absolute attention of all as the 2×2 and 3×3 matrices are the most common matrices employed in texts and by students and teachers of mathematics, science and engineering. Thus, the Author strongly recommend this new method for hand–computation of inverses of matrices worldwide.

Let us now compute the inverse of the 3×3 matrix,

$$\mathbf{D} = \begin{bmatrix} 0 & 3 & 0 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}.$$

By rule 1 we write the four-quadrant matrix as

$$\begin{bmatrix} 0 & 3 & 0 & 0 & 3 & 0 \\ 4 & 1 & 6 & 4 & 1 & 6 \\ 1 & 4 & 0 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 4 & 1 & 6 & 4 & 1 & 6 \\ 1 & 4 & 0 & 1 & 4 & 0 \end{bmatrix},$$

and by applying rule 2 we get the 4×4 interior matrix of the four-quadrant matrix as

$$\begin{bmatrix} 1 & 6 & 4 & 1 \\ 4 & 0 & 1 & 4 \\ 3 & 0 & 0 & 3 \\ 1 & 6 & 4 & 1 \end{bmatrix},$$

which, by rule 3, becomes the cofactor matrix ,

$$\begin{bmatrix} -24 & 6 & 15 \\ 0 & 0 & 3 \\ 18 & 0 & -12 \end{bmatrix}$$

which in its turn, after being transposed, becomes

$$\begin{bmatrix} -24 & 0 & 18 \\ 6 & 0 & 0 \\ 15 & 3 & -12 \end{bmatrix}.$$

We stop the condensation process and start the computation of the determinant of **D**. Since, of the four elements in the interior of the 4×4 interior matrix of the four-quadrant matrix, only one element is non–zero, there is, therefore, only one way of finding the determinant of **D** from the cofactor matrix, namely, the evaluation in which the divisor is the element, 1. Thus the determinant of **D** is

$$|\mathbf{D}| = \frac{1}{1}\begin{vmatrix} 6 & 15 \\ 0 & 3 \end{vmatrix} = 18.$$

and its inverse is

$$\mathbf{D^{-1}} = \frac{1}{18}\begin{bmatrix} -24 & 0 & 18 \\ 6 & 0 & 0 \\ 15 & 3 & -12 \end{bmatrix}.$$

We now turn to the computation of the inverse of the 4×4 matrix,

$$\begin{bmatrix} -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 \end{bmatrix}.$$

We write the four–quadrant matrix as

$$\begin{bmatrix} -2 & 3 & 4 & 7 & -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 & 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 & 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 & 2 & 1 & 4 & 2 \\ -2 & 3 & 4 & 7 & -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 & 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 & 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 & 2 & 1 & 4 & 2 \end{bmatrix}$$

and obtain its interior matrix as

$$\begin{bmatrix} 2 & 4 & 4 & 6 & 2 & 4 \\ -3 & 6 & 3 & 3 & -3 & 6 \\ 1 & 4 & 2 & 2 & 1 & 4 \\ 3 & 4 & 7 & -2 & 3 & 4 \\ 2 & 4 & 4 & 6 & 2 & 4 \\ -3 & 6 & 3 & 3 & -3 & 6 \end{bmatrix}.$$

We employ Dodgson's condensation and obtain the following:

$$\begin{bmatrix} 24 & -12 & -6 & -24 & 24 \\ -18 & 0 & 0 & 9 & -18 \\ -8 & 20 & -18 & 8 & -8 \\ 4 & -12 & 50 & -22 & 4 \\ 24 & -12 & 50 & -22 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -36 & 0 & -18 & -72 \\ -90 & 0 & 81 & 72 \\ 4 & 112 & 2 & -48 \\ 60 & 168 & -222 & -216 \end{bmatrix}.$$

We give the elements of this matrix their prescribed signs since the original matrix is of even order, and we obtain the cofactor matrix as

$$\begin{bmatrix} -36 & 0 & -18 & 72 \\ 90 & 0 & -81 & 72 \\ 4 & -112 & 2 & 48 \\ -60 & 168 & 222 & -216 \end{bmatrix}.$$

The determinant of $\mathbf{D}$ is

$$|\mathbf{D}| = \frac{1}{9}\begin{vmatrix} -18 & -72 \\ 81 & 72 \end{vmatrix} = 504.$$

and its inverse is

$$\mathbf{D^{-1}} = \frac{1}{504}\begin{bmatrix} -36 & 90 & 4 & -60 \\ 0 & 0 & -112 & 168 \\ -18 & -81 & 2 & 222 \\ 72 & 72 & 48 & -216 \end{bmatrix}.$$

This method is exceedingly simple and lucid, but it may be rendered even more palpable to the eye by arranging the series of matrices one under another, as it is displayed below; it will then be found very easy to culled the divisors from the matrices:

$$\begin{bmatrix} -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & 4 & 7 & -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 & 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 & 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 & 2 & 1 & 4 & 2 \\ -2 & 3 & 4 & 7 & -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 & 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 & 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 & 2 & 1 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 4 & 6 & 2 & 4 \\ -3 & 6 & 3 & 3 & -3 & 6 \\ 1 & 4 & 2 & 2 & 1 & 4 \\ 3 & 4 & 7 & -2 & 3 & 4 \\ 2 & 4 & 4 & 6 & 2 & 4 \\ -3 & 6 & 3 & 3 & -3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 24 & -12 & -6 & -24 & 24 \\ -18 & 0 & 0 & 9 & -18 \\ -8 & 20 & -18 & 8 & -8 \\ 4 & -12 & 50 & -22 & 4 \\ 24 & -12 & 50 & -22 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -36 & 0 & -18 & -72 \\ -96 & 0 & 81 & 72 \\ 4 & 112 & 2 & -48 \\ 60 & 168 & -222 & -216 \end{bmatrix}.$$

We take the computation of the inverse of another 4×4 matrix. This is to teach us how we may handle cases wherein zero appears as a divisor. Suppose the matrix

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 5 \end{bmatrix}.$$

We work as follows:

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 1 & 2 & 3 & -1 \\ 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 & 2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 5 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & -1 & 1 & 2 & 3 & -1 \\ 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 & 2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 5 & 1 & 2 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 2 & 1 & -1 \\ 2 & 3 & 5 & 1 & 2 & 3 \\ 2 & 3 & -1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 2 & 1 & -1 \end{bmatrix}.$$

We add, based on Bhaskara's law of zero, the zero $\mathbf{0}$ to the interior zero and apply condensation as usual:

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 2 & 1 & -1 \\ 2 & 3 & 5 & 1 & 2 & 3 \\ 2 & 3 & -1 & 1 & 2 & 3 \\ 1 & \mathbf{0} & 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 2 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 \\ 5 & -8 & -9 & 3 & 5 \\ 0 & -18 & 6 & 0 & 0 \\ 2\cdot\mathbf{0}-3 & 3+\mathbf{0} & -3 & -3 & -3 \\ -1-\mathbf{0} & \mathbf{0}+1 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -9 & 0 & 3 \\ -30 & -42 & -18 & 0 \\ 12\cdot\mathbf{0}-18 & 6\cdot\mathbf{0}-36 & -18 & 0 \\ 2\cdot\mathbf{0}+3 & 3\cdot\mathbf{0}+3 & 0 & 3 \end{bmatrix}.$$

We omit $\mathbf{0}$ since it represents nothing. Thus, we have the matrix

$$\begin{bmatrix} -3 & -9 & 0 & 3 \\ -30 & -42 & -18 & 0 \\ -18 & -36 & -18 & 0 \\ 3 & 3 & 0 & 3 \end{bmatrix}$$

which, giving its elements their prescribed signs, becomes

$$\begin{bmatrix} -3 & 9 & 0 & -3 \\ 30 & -42 & 18 & 0 \\ -18 & 36 & -18 & 0 \\ -3 & 3 & 0 & 3 \end{bmatrix}.$$

Thus the inverse is

$$\frac{1}{18}\begin{bmatrix} -3 & 30 & -18 & -3 \\ 9 & -42 & 36 & 3 \\ 0 & 18 & -18 & 0 \\ -3 & 0 & 0 & 3 \end{bmatrix}.$$

4.1 Proof of the Validity of the New Approach

We now proceed to give a proof of the validity of this new method. In doing so, we shall take the following steps in computing the cofactor matrix of the $n \times n$ matrix $\mathbf{D}$ by means of the new method:

1. Form the cofactor matrix consisting of the cofactors or minors in determinant form.
2. Rearrange the elements in the determinants, the minors, such that
   (a) for every row, from left to right, the 2nd, 3rd, 4th, ..., $(n-1)$st columns of each determinant are the respective 1st, 2nd, 3rd, ..., $(n-2)$nd columns of the next determinant.
   (b) for every column, from top to bottom, the 2nd, 3rd, 4th, ..., $(n-1)$st rows of each determinant are the respective 1st, 2nd, 3rd, ..., $(n-2)$nd rows of the next determinant.
3. Write all the columns of the elements of all the determinants without repeating any column.

If these steps are carefully taken, it will be found that the new matrix formed is the interior of a matrix formed by putting $\mathbf{D}$ in four quadrants, adjacent to one another.

4.1.1 Derivation for 2×2 Matrix

Let us first take the simplest case, the 2×2 matrix:

$$\mathbf{D} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

If we compute the cofactors of the elements of this matrix by the method of finding the complementary minor of each element, we obtain the cofactor matrix

$$\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

which, except for the prescribed signs of the cofactors, is the interior of the matrix,

$$\begin{bmatrix} a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \\ a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \end{bmatrix},$$

formed by putting $\mathbf{D}$ in four quadrants, adjacent to one another.

4.1.2 Derivation for 3×3 Matrix

Secondly, let us take the 3×3 matrix:

$$\mathbf{D} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

We find the cofactor matrix by method of minors and get

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}.$$

We rearrange the elements of each determinant in the cofactor matrix above, such that for every row, from left to right, the 2nd column of each determinant is the 1st column of the next determinant. Thus we have the cofactor matrix of **D** rearranged as

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{13} & a_{11} \\ a_{33} & a_{31} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}.$$

Again, we rearrange the elements of each determinant, such that for every column, from top to bottom, the 2nd row of each determinant is the 1st row of the next determinant. So we have the cofactor matrix rewritten as

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}.$$

This new arrangement of the cofactor matrix may be considered as a derived matrix obtained by employing Dodgson's condensation to the matrix,

$$\begin{bmatrix} a_{22} & a_{23} & a_{21} & a_{22} \\ a_{32} & a_{33} & a_{31} & a_{32} \\ a_{12} & a_{13} & a_{11} & a_{12} \\ a_{22} & a_{23} & a_{21} & a_{22} \end{bmatrix},$$

which is clearly the interior of the matrix,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \end{bmatrix},$$

formed by putting **D** in four quadrants, adjacent to one another. This proves the method for a 3×3 matrix; and similar proofs might be given for larger matrices.

4.1.3 Derivation for 4×4 Matrix

Lastly, let us take the 4×4 matrix:

$$\mathbf{D} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

We find the cofactor matrix by method of minors and get

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{bmatrix}.$$

We rearrange the elements of each determinant in the cofactor matrix above, such that for every row, from left to right, the 2nd and 3rd columns of each determinant are respectively the 1st and 2nd columns of the next determinant. Thus we have the cofactor matrix of **D** rearranged as

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} \\ a_{43} & a_{44} & a_{41} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{21} & a_{22} \\ a_{24} & a_{31} & a_{32} \\ a_{24} & a_{41} & a_{42} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} & a_{11} \\ a_{33} & a_{34} & a_{31} \\ a_{43} & a_{44} & a_{41} \end{vmatrix} & -\begin{vmatrix} a_{14} & a_{11} & a_{12} \\ a_{34} & a_{31} & a_{32} \\ a_{44} & a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{13} & a_{14} & a_{11} \\ a_{23} & a_{24} & a_{21} \\ a_{43} & a_{44} & a_{41} \end{vmatrix} & \begin{vmatrix} a_{14} & a_{11} & a_{12} \\ a_{24} & a_{21} & a_{22} \\ a_{44} & a_{41} & a_{42} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} & a_{11} \\ a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} \end{vmatrix} & -\begin{vmatrix} a_{14} & a_{11} & a_{12} \\ a_{24} & a_{21} & a_{12} \\ a_{34} & a_{31} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{bmatrix}.$$

Again, we rearrange the elements of each determinant, such that for every column, from top to bottom, the 2nd and 3rd rows of each determinant are respectively the 1st and 2nd rows of the next determinant. So we have the cofactor matrix rewritten as

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} \\ a_{43} & a_{44} & a_{41} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{21} & a_{22} \\ a_{34} & a_{31} & a_{32} \\ a_{44} & a_{41} & a_{42} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\ -\begin{vmatrix} a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{12} & a_{13} & a_{14} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} & a_{31} \\ a_{43} & a_{44} & a_{41} \\ a_{13} & a_{14} & a_{11} \end{vmatrix} & -\begin{vmatrix} a_{34} & a_{31} & a_{32} \\ a_{44} & a_{41} & a_{42} \\ a_{14} & a_{11} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} & a_{44} \\ a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \end{vmatrix} & -\begin{vmatrix} a_{43} & a_{44} & a_{41} \\ a_{13} & a_{14} & a_{11} \\ a_{23} & a_{24} & a_{21} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{41} & a_{42} \\ a_{14} & a_{11} & a_{12} \\ a_{24} & a_{21} & a_{22} \end{vmatrix} & -\begin{vmatrix} a_{41} & a_{42} & a_{43} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} & a_{11} \\ a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} \end{vmatrix} & -\begin{vmatrix} a_{14} & a_{11} & a_{12} \\ a_{24} & a_{21} & a_{22} \\ a_{34} & a_{31} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{bmatrix}.$$

This new arrangement of the cofactor matrix may be considered, removing the prescribed signs, as a derived matrix obtained by employing Dodgson's condensation to the matrix,

$$\begin{bmatrix} a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} \\ a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} \\ a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{43} \\ a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} \\ a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} \\ a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} \end{bmatrix},$$

which is clearly the interior of the matrix,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{43} & a_{44} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{44} & a_{11} & a_{12} & a_{13} & a_{44} \end{bmatrix},$$

formed by putting **D** in four quadrants, adjacent to one another. This proves the method for a 4×4 matrix; and similar proofs might be given for larger matrices.

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Author has declared that no competing interests exist.

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